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EDITED BY

E. T. BELL
CALIFORNIA INSTITUTE OF TECHNOLOGY

ABRAHAM COHEN
THE JOHNS HOPKINS UNIVERSITY

E. W. CHITTENDEN
UNIVERSITY OF IOWA

G. C. EVANS
UNIVERSITY OF CALIFORNIA

F. D. MURNAGHAN
THE JOHNS HOPKINS UNIVERSITY

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POSTULATES FOR BOOLEAN ALGEBRAS AND GENERALIZED BOOLEAN ALGEBRAS.

By M. H. STONE.

In this paper we present a new set of postulates for Boolean algebras in terms of logical addition and logical multiplication (or, to use the language of the algebra of classes, in terms of union and intersection), a similar set of postulates for generalized Boolean algebras (distinguished by the possible absence of a unit or "universe" element), and an investigation of the independence of the proposed postulates. This material was originally designed to constitute an introduction to a general theory of Boolean algebras, to be developed by the methods appropriate to modern concepts of algebra.[1] The postulates were therefore chosen with the intention of emphasizing the analogy between Boolean algebras and abstract rings, the latter being systems which have already undergone extensive analysis. From this point of view our postulates appear to be as satisfactory as possible, so long as logical addition and multiplication are to be treated as the analogues of ring addition and multiplication. We have recently observed, however, that Boolean algebras can be regarded as rings of special type when the operation of forming the symmetric difference is taken as ring addition.[2] In consequence, the most natural approach to a mathematical theory of Boolean algebras is not one based upon the material of the present paper. Since the results which are collected here seem to possess some intrinsic interest—in particular, they offer a simplification of Huntington's set of postulates in terms of logical addition and multiplication, and also establish the redundance of one of the postulates in the set proposed by Del Re [3]—we venture to present them apart from our general theory of Boolean algebras.

1. *A new set of postulates for Boolean algebras.* We shall define a Boolean algebra as a system with double composition in which the operations

[1] For a brief sketch of the theory, see *Bulletin of the American Mathematical Society*, abstract No. 39-3-86; *Proceedings of the National Academy of Sciences*, vol. 20 (1934), pp. 197-202.

[2] See *Proceedings of the National Academy of Sciences*, vol. 21 (1935), pp. 103-105; the formal definition of the symmetric difference in terms of logical addition and multiplication is given as Definition 4 below.

[3] See § 2 below.

of addition and multiplication are required to satisfy certain further postulates. In selecting these postulates we are guided by three principles: in the first place, they shall embody known properties of the operations of forming finite unions and intersections of classes; in the second place, they shall embody so far as possible only such properties as are valid of the operations of addition and multiplication in a ring; and, in the third place, they shall be independent. We recall [4] that a ring is a system with double composition in which addition is commutative, both addition and multiplication are associative, multiplication is both left-distributive and right-distributive with respect to addition, and the equation [5] $x \vee a = b$ has a solution for arbitrary elements a and b. It is easily shown that in an arbitrary ring this equation has a unique solution, and that there exists a unique element 0 such that $a \vee 0 = 0 \vee a = a$ for every element a. In an algebra of classes, the operations of forming the finite union and finite intersection have all these properties except that the equation $\mathfrak{x} \cup \mathfrak{a} = \mathfrak{b}$ does not have a solution unless $\mathfrak{a} \subset \mathfrak{b}$ and does not have a unique solution, even when this condition is satisfied, unless the auxiliary relation $\mathfrak{x}\mathfrak{a} = \mathfrak{o}$ be required. The algebra of classes is further differentiated by the existence of the special properties expressed by the equations $\mathfrak{a} \cup \mathfrak{a} = \mathfrak{a}$ and $\mathfrak{a}\mathfrak{a} = \mathfrak{a}$. The second of these equations is not generally true for rings but can be verified without exception in certain special rings. On the other hand, the only ring element which has the property that $a \vee a = a$ is the element 0 described above. In accordance with the first two principles stated above, we may postulate the commutative law for addition, the associative law for addition and multiplication, the two distributive laws for multiplication with respect to addition, the existence of an element 0 such that $a \vee 0 = a$ for every element a, the existence of a solution of the simultaneous equations $x \vee a = e$ and $xa = 0$ for every element a and a certain fixed element e, and the two special rules $a \vee a = a$ and $aa = a$. Every algebra of classes has all these properties; there are special rings which have all these properties except the one expressed by the equation $a \vee a = a$; and the only ring which has all these properties is the trivial ring consisting of the single element 0. Our third principle, however, requires us to reject the associative laws for addition and multiplication, since they can be deduced from the other properties listed. Thus our definition of Boolean algebras can be stated as follows:

DEFINITION 1. *A Boolean algebra A is a system with double composition which satisfies the further postulates:*

---

[4] See B. L. van der Waerden, *Moderne Algebra*, Bd. 1 (Berlin, 1930), pp. 36-39.

[5] We shall denote addition by the symbol $\vee$.

Postulate 1_1. $a \vee b = b \vee a$;

Postulate 3_1. $a(b \vee c) = ab \vee ac$;

Postulate 3_2. $(a \vee b)c = ac \vee bc$;

Postulate 4_1. *There exists an element* 0 *in* A *such that* $a \vee 0 = a$ *for every element* a *in* A;

* *Postulate* 5. *If there exists an element* 0 *with the property required by Postulate* 4_1, *there exists at least one such element* 0 *to which corresponds a fixed element* e *in* A *such that the simultaneous equations* $x \vee a = e$, $xa = 0$ *have a solution for every element* a *in* A;

Postulate 6_1. $a \vee a = a$;

Postulate 6_2. $aa = a$.

Any element such as that postulated in Postulate 4_1 *is called a zero; and any element such as the element* e *postulated in Postulate* 5 *is called a unit.*

The numbering of the postulates has been chosen with a view to certain subsequent modifications in the system of postulates. We may point out that, in accordance with Postulate 4_1, a Boolean algebra contains at least one element. It is often required that a Boolean algebra have at least two elements; but for our purposes it is preferable to admit as a Boolean algebra a one-element system, consisting of zero elements alone. We call attention to the fact that the replacement of the simultaneous equations of Postulate 5 by the simultaneous equations $x \vee a = e$, $ax = 0$ does not constitute an essential alteration: if we rewrite the altered set of postulates in terms of a new multiplication o defined by writing $a \circ b$ for ba we obtain the original set once more.

We must now proceed to deduce from the postulated properties of the operations in a Boolean algebra a series of further properties, including, for instance, the associative law for addition and multiplication and the commutative law for multiplication. In order to condense the demonstrations, we shall arrange them so far as possible in the form of continued equations, citing the propositions used in passing across successive equality signs by successive groups set off by commas. In the citations, we shall replace the words postulate, definition, theorem, lemma, and hypothesis by their initial letters P, D, T, L, and H respectively. Those propositions which do not hold in generalized Boolean algebras, introduced later, will be indicated by prefixing an asterisk.

THEOREM 1. *The element* 0 *of Postulate* 4_1 *is unique.*

If 0_1 and 0_2 are elements with the property postulated in P 4_1, then $0_1 = 0_1 \vee 0_2 = 0_2 \vee 0_1 = 0_1$ by P 4_1, P 1_1, P 4_1.

* THEOREM 2. *If* 0 *and* e *are elements with the properties demanded in* * *Postulate* 5, *then* $ea = a$ *for every element* a.

Let x be an element such that $x \vee a = e$, $xa = 0$ in accordance with * P 5. Then by *P 5, P 3_2, *P 5 — P 6_2, P 1_1, P 4_1 we have $ea = (x \vee a)a = xa \vee aa = 0 \vee a = a \vee 0 = a$.

* LEMMA 1. *If* a' *is any solution of the equations* $x \vee a = e$, $xa = 0$, *then* $a'e = a'$.

By * P 5, P 3_1, P 6_2 — * P 5, P 4_1, we have $a'e = a'(a' \vee a) = a'a' \vee a'a = a' \vee 0 = a'$.

* LEMMA 2. *If* a' *is any solution of the equations* $x \vee a = e$, $xa = 0$, *and if* a'' *is any solution of the equations* $x \vee a' = e$, $xa' = 0$, *then* $a'' = a$.

By *L 1, *P 5, P 3_1, *P 5, *P 5, P 3_2, P 1_1, *P 5, *T 2, we have

$$a'' = a''e = a''(a' \vee a) = a''a' \vee a''a = 0 \vee a''a$$
$$= a'a \vee a''a = (a' \vee a'')a = (a'' \vee a')a = ea = a.$$

* LEMMA 3. *If* a' *is any solution of the equations* $x \vee a = e$, $xa = 0$, *then* $aa' = 0$.

By *L 2, we have $aa' = a''a' = 0$.

* THEOREM 3. *If* 0 *and* e *are elements with the properties demanded in* * *Postulate* 5, *then* $ae = a$ *for every element* a.

Let a' be any solution of the equations $x \vee a = e$, $xa = 0$. *Then by* *P 5, P 3_1, *L 3 — P 6_2, P 1_1, P 4_1 we have $ae = a(a' \vee a) = aa' \vee aa = 0 \vee a = a \vee 0 = a$.

* THEOREM 4. *The element* e *of Postulate* 5 *is unique.*

If e_1 and e_2 are any two elements with the property postulated in *P 5, whether they correspond to the same or equal zero elements, we have $e_1 = e_1e_2 = e_2$ by *T 3, *T 2.

* THEOREM 5. *If* e *is the element described in* * *Postulate* 5, *then* $e \vee a = a \vee e = e$ *for every element* a.

Let a' be any solution of the equations $x \vee a = e$, $xa = 0$. Then, by P 1_1, *T 2, *P 5, P 3_2, P 3_1, *P 5 — *T 4 — P 6_2 — *T 4, P 1_1 — P 4_1 — P 6_1, *P 5, we have

$$e \vee a = a \vee e = e(a \vee e) = (a' \vee a)(a \vee e) = a'(a \vee e) \vee a(a \vee e)$$
$$= (a'a \vee a'e) \vee (aa \vee ae) = (0 \vee a') \vee (a \vee a) = a' \vee a = e.$$

THEOREM 6. *If 0 is the element of Postulate 4, then $a0 = 0a = 0$ for every element a.*

By P 4_1, *T 2, P 3_2, *T 5, *T 2, we have $a0 = a0 \vee 0 = a0 \vee e0 = (a \vee e)0 = e0 = 0$; and, by P 4_1, *T 3, P 3_1, *T 5, *T 3, we have $0a = 0a \vee 0 = 0a \vee 0e = 0(a \vee e) = 0e = 0$.

* THEOREM 7. *The equations $x \vee a = e$, $xa = 0$ of * Postulate 5 have a unique solution.*

Let 0_1, 0_2 be zero elements; let e_1 and e_2 be corresponding unit elements; let a'_1 be any solution of the equations $x \vee a_1 = e_1$, $xa_1 = 0_1$; let a'_2 be any solution of the equations $x \vee a_2 = e_2$, $xa_2 = 0_2$; and let $a_1 = a_2$. By *T 3, *P 5, H, P 3_1, *P 5, T 1, *L 3, P 3_2, H, *P 5, *T 2, we then have

$$a'_1 = a'_1 e_2 = a'_1(a'_2 \vee a_2) = a'_1(a'_2 \vee a_1) = a'_1 a'_2 \vee a'_1 a_1 = a'_1 a'_2 \vee 0_1$$
$$= a'_1 a'_2 \vee 0_2 = a'_1 a'_2 \vee a_2 a'_2 = (a'_1 \vee a_2)a'_2 = (a'_1 \vee a_1)a'_2 = e_1 a'_2 = a'_2.$$

In view of * Theorem 7, * Lemma 2, it is convenient to introduce the following definition and theorem.

* DEFINITION 2. *The unary operation which takes the element a into the solutions of the equations of * Postulate 5, namely, the simultaneous equations $x \vee a = e$ and $xa = 0$, is called complementation; and the element which results from this operation is denoted by a' and is called the complement of a. The element $(a')'$ is denoted by a''.*

* THEOREM 8. *The operation of complementation has the following propetries:*

(1) *$a = b$ implies $a' = b'$;* (2) *$a'' = a$;* (3) *$0' = e$;* (4) *$e' = 0$.*

This theorem merely summarizes the results of * Theorem 7 and * Lemma 2 and the special cases $e \vee 0 = e$, $e0 = 0$ and $0 \vee e = e$, $0e = 0$ of the relations proved in * Theorems 5 and 6.

* LEMMA 4_1. *If $a \vee b = 0$, then $a'b = 0$.*

By P 4_1, *P 5, P 3_1 — P 1_1, H, T 6, we have

$$a'b = a'b \vee 0 = a'b \vee a'a = a'(a \vee b) = a'0 = 0.$$

* LEMMA 4_2. *If* $a \vee b = 0$, *then* $ab' = 0$.

By P 4_1, *L 3, P 3_2, H, T 6, we have

$$ab' = ab' \vee 0 = ab' \vee bb' = (a \vee b)b' = 0b' = 0.$$

THEOREM 9. *If* $a \vee b = 0$, *then* $a = b = 0$.

By P 1_1 and T 1, it is sufficient to show that $a = 0$. By P 6_1, *T 3, *P 5, P 3_1, *L 4_2, *L 4_1, P 3_2, *P 5, *T 2, H, we have

$$\begin{aligned} a = a \vee a = a \vee ae = a \vee a(b' \vee b) = a \vee (ab' \vee ab) \\ = a \vee (0 \vee ab) = a \vee (a'b \vee ab) = a \vee (a' \vee a)b = a \vee eb = a \vee b = 0. \end{aligned}$$

* LEMMA 5_1. *If* $c = (a \vee b)'$, *then* $ca = 0$.

By P 3_1, *P 5, we have $ca \vee cb = c(a \vee b) = 0$. By T 9, we then have $ca = 0$.

* LEMMA 5_2. *If* $c = (a \vee b)'$, *then* $ac = 0$.

By P 3_2, *L 3, we have $ac \vee bc = (a \vee b)c = 0$. By T 9, we then have $ac = 0$.

* LEMMA 5_3. *If* $c = (a \vee b)'$, *then* $ca' = c$.

By P 4_1, *L 5_1, P 3_1, *P 5, *T 3, we have

$$ca' = ca' \vee 0 = ca' \vee ca = c(a' \vee a) = ce = c.$$

* LEMMA 5_4. *If* $c = (a \vee b)'$, *then* $a'c = c$.

By P 4_1, *L 5_2, P 3_2, *P 5, *T 2, we have

$$a'c = a'c \vee 0 = a'c \vee ac = (a' \vee a)\, c = ec = c.$$

* LEMMA 5_5. *If* $c = (a \vee b)'$, *then* $a' = c \vee a'b$.

By *T 3, *P 5, P 3_1, *L 5_4 — P 3_1, *P 5, P 1_1, P 4_1, we have

$$\begin{aligned} a' = a'e = a'[c \vee (a \vee b)] = a'c \vee a'(a \vee b) \\ = c \vee (a'a \vee a'b) = c \vee (0 \vee a'b) = c \vee (a'b \vee 0) = c \vee a'b. \end{aligned}$$

* LEMMA 5_6. *If* $c = (a \vee b)'$, *then* $a' = c \vee ba'$.

By *T 2, *P 5, P 3_2, *L 5_3 — P 3_2, *L 3, P 1_1, P 4_1, we have

$$\begin{aligned} a' = ea' = [c \vee (a \vee b)]a' = ca' \vee (a \vee b)a' \\ = c \vee (aa' \vee ba') = c \vee (0 \vee ba') = c \vee (ba' \vee 0) = c \vee ba'. \end{aligned}$$

* LEMMA 6_1. $a(a'b) = 0$.

Let $c = (a \vee b)'$. Then, by P 4_1, *L 5_2, P 3_1, P 1_1, *L 5_5, *L 3, we have

$$a(a'b) = a(a'b) \vee 0 = a(a'b) \vee ac = a(a'b \vee c) = a(c \vee a'b) = aa' = 0.$$

* LEMMA 6_2. $a(ba') = 0$.

Let $c = (a \vee b)'$. Then, by P 4_1, *L 5_2, P 3_1, P 1_1, *L 5_6, *L 3, we have

$$a(ba') = a(ba') \vee 0 = a(ba') \vee ac = a(ba' \vee c) = a(c \vee ba') = aa' = 0.$$

* LEMMA 6_3. $(a'b)a = 0$.

Let $c = (a \vee b)'$. Then, by P 4_1, *L 5_1, P 3_2, P 1_1, *L 5_5, *P 5, we have

$$(a'b)a = (a'b)a \vee 0 = (a'b)a \vee ca = (a'b \vee c)a = (c \vee a'b)a = a'a = 0.$$

* LEMMA 6_4. $(ba')a = 0$.

Let $c = (a \vee b)'$. Then, by P 4_1, *L 5_1, P 3_2, P 1_1, *L 5_6, *P 5, we have

$$(ba')a = (ba')a \vee 0 = (ba')a \vee ca = (ba' \vee c)a = (c \vee ba')a = a'a = 0.$$

THEOREM 10. $a \vee ab = a \vee ba = a$.

Let $c = (a \vee b)'$. Then, by P 6_2 — P 4_1, P 3_1 — *L 5_2, P 3_1, P 1_1, *P 5, *T 3, *T 2, *P 5, P 3_2, *L 5_1 — P 3_2, P 6_2 — P 1_1, P 4_1, we have

$$a \vee ab = (aa \vee ab) \vee 0 = a(a \vee b) \vee ac = a[(a \vee b) \vee c] = a[c \vee (a \vee b)] = ae$$
$$= a = ea = [c \vee (a \vee b)]a = ca \vee (a \vee b)a = 0 \vee (aa \vee ba) = (a \vee ba) \vee 0 = a \vee ba.$$

* LEMMA 7. $a \vee a'b = a \vee b$.

By T 10 — P 4_1, P 1_1, P 6_2 — *P 5, P 3_1, P 3_1, P 1_1, *P 5, *T 2, we have

$$a \vee a'b = (a \vee ab) \vee (a'b \vee 0) = (a \vee ab) \vee (0 \vee a'b) = (aa \vee ab) \vee (a'a \vee a'b)$$
$$= a(a \vee b) \vee a'(a \vee b) = (a \vee a')(a \vee b) = (a' \vee a)(a \vee b) = e(a \vee b) = a \vee b.$$

LEMMA 8_1. $(a \vee b)(a \vee c) = a \vee (ba \vee bc)$.

By P 3_2, P 3_1, P 6_2, T 10, we have

$$(a \vee b)(a \vee c) = a(a \vee c) \vee b(a \vee c)$$
$$= (aa \vee ac) \vee (ba \vee bc) = (a \vee ac) \vee (ba \vee bc) = a \vee (ba \vee bc).$$

LEMMA 8_2. $a[a \vee (ba \vee bc)] = a$.

By P 3_1, P 6_2, T 10, we have

$$a[a \vee (ba \vee bc)] = aa \vee a(ba \vee bc) = a \vee a(ba \vee bc) = a.$$

* LEMMA 8_3. $a'[a \vee (ba \vee bc)] = a'(bc)$.

By P 3_1, *P 5, P 1_1, P 4_1 — P 3_1, *L 2, *L 6_2, P 1_1, P 4_1, we have

$$a'[a \vee (ba \vee bc)] = a'a \vee a'(ba \vee bc) = 0 \vee a'(ba \vee bc) = a'(ba \vee bc) \vee 0$$
$$= a'(ba) \vee a'(bc) = a'(ba'') \vee a'(bc) = 0 \vee a'(bc) = a'(bc) \vee 0 = a'(bc).$$

THEOREM 11. $(a \vee b)(a \vee c) = a \vee bc$.

By L 8_1, *T 2, *P 5, P 3_2, *L 8_3 — L 8_2, P 1_1, *L 7, we have

$$(a \vee b)(a \vee c) = a \vee (ba \vee bc) = e[a \vee (ba \vee bc)] = (a' \vee a)[a \vee (ba \vee bc)]$$
$$= a'[a \vee (ba \vee bc)] \vee a[a \vee (ba \vee bc)] = a'(bc) \vee a = a \vee a'(bc) = a \vee bc.$$

* LEMMA 9. $(a \vee b) \vee a' = e$.

By *T 2, *P 5, P 3_2, P 3_1, P 3_1 — P 6_2 — P 3_1 — *L 3, *P 5 — P 6_2 — P 4_1, P 1_1 — T 10, P 4_1, P 1_1, T 10, *P 5, we have

$$(a \vee b) \vee a' = e[(a \vee b) \vee a'] = (a' \vee a)[(a \vee b) \vee a']$$
$$= a'[(a \vee b) \vee a'] \vee a[(a \vee b) \vee a'] = [a'(a \vee b) \vee a'a'] \vee [a(a \vee b) \vee aa']$$
$$= [(a'a \vee a'b) \vee a'] \vee [(aa \vee ab) \vee 0] = [(0 \vee a'b) \vee a'] \vee (a \vee ab)$$
$$= [(a'b \vee 0) \vee a'] \vee a = (a'b \vee a') \vee a = (a' \vee a'b) \vee a = a' \vee a = e.$$

* THEOREM 12_1. $(a \vee b)' = a'b'$.

By *T 7, it is sufficient to prove that $a'b'$ is a solution of the simultaneous equations $x \vee (a \vee b) = e$, $x(a \vee b) = 0$. By P 1_1, T 11, *L 9 — P 1_1, *L 9, P 6_2, we have

$$a'b' \vee (a \vee b) = (a \vee b) \vee a'b' = [(a \vee b) \vee a'][(a \vee b) \vee b'] = e[(b \vee a) \vee b'] = ee = e;$$

and, by P 3_1, *L 6_3 — *L 6_4, P 6_1, we have also

$$a'b'(a \vee b) = (a'b')a \vee (a'b')b = 0 \vee 0 = 0.$$

* THEOREM 12_2. *$ab = (a' \vee b')'$ and $(ab)' = a' \vee b'$.*

By *L 2, *T 12_1, we have $ab = a''b'' = (a' \vee b')'$; and, by *L 2, $(ab)' = (a' \vee b')'' = a' \vee b'$.

THEOREM 13. $ba = ab$.

By *T 12_2, P 1_1, *T 12_2, we have $ba = (b' \vee a')' = (a' \vee b')' = ab$.

LEMMA 10_1. $a[(a \vee b) \vee c] = a[a \vee (b \vee c)] = a$.

By P 3_1, P 3_1, P 6_2, T 10, T 10, T 10, P 6_2, P 3_1, we have

$$a[(a \vee b) \vee c] = a(a \vee b) \vee ac = (aa \vee ab) \vee ac = (a \vee ab) \vee ac$$
$$= a \vee ac = a = a \vee a(b \vee c) = aa \vee a(b \vee c) = a[a \vee (b \vee c)].$$

* LEMMA 10_2. $a'[(a \vee b) \vee a] = a'[a \vee (b \vee c)] = a'(b \vee c)$.

By P 3_1, P 3_1, *P 5, P 1_1, P 4_1, P 3_1, P 4_1, P 1_1, *P 5, P 3_1, we have

$$a'[(a \vee b) \vee c] = a'(a \vee b) \vee a'c = (a'a \vee a'b) \vee a'c = (0 \vee a'b) \vee a'c$$
$$= (a'b \vee 0) \vee a'c = a'b \vee a'c = a'(b \vee c) = a'(b \vee c) \vee 0$$
$$= 0 \vee a'(b \vee c) = a'a \vee a'(b \vee c) = a'[a \vee (b \vee c)].$$

THEOREM 14. $(a \vee b) \vee c = a \vee (b \vee c)$.

By *T 2, *P 5, P 3_2, L 10_1 — *L 10_2, P 3_2, *P 5, *T 2, we have

$$(a \vee b) \vee c = e[(a \vee b) \vee c] = (a' \vee a)[(a \vee b) \vee c]$$
$$= a'[(a \vee b) \vee c] \vee a[(a \vee b) \vee c] = a'[a \vee (b \vee c)] \vee a[a \vee (b \vee c)]$$
$$= (a' \vee a)[a \vee (b \vee c)] = e[a \vee (b \vee c)] = a \vee (b \vee c).$$

THEOREM 15. $(ab)c = a(bc)$.

By *T 12_2, *T 12_2, T 14, *T 12_2, *T 12_2, we have

$$(ab)c = ((ab)' \vee c')' = ((a' \vee b') \vee c')' = (a' \vee (b' \vee c'))' = (a' \vee (bc)')' = a(bc).$$

THEOREM 16_1. *If the simultaneous equations $x \vee a = b$, $xa = 0$ have a solution c, then $ba = a$, $bc = c$, and the solution is unique.*

If c is a solution, we have $ba = (c \vee a)a = ca \vee aa = 0 \vee a = a \vee 0 = a$ by H, P 3_2, H — P 6_2, P 1_1, P 4_1; and also $bc = (c \vee a)c = cc \vee ca = c \vee 0 = c$ by H, P 3_2 — T 13, P 6_2 — H, P 4_1. If c_1 and c_2 are solutions of the respective systems $x \vee a_1 = b_1$, $xa_1 = 0_1$ and $x \vee a_2 = b_2$, $xa_2 = 0_2$, where $a_1 = a_2$, $b_1 = b_2$, and $0_1 = 0_2$, we have to show that $c_1 = c_2$. For this it is sufficient, by what we have just proved, to show that $b_1c_1 = b_2c_2$. Now we have

$$b_1c_1 = b_2c_1 = (c_2 \vee a_2)c_1 = (c_2 \vee a_1)c_1 = c_2c_1 \vee a_1c_1 = c_1c_2 \vee c_1a_1$$
$$= c_1c_2 \vee 0_1 = c_1c_2 \vee 0_2 = c_1c_2 \vee c_2a_2 = c_1c_2 \vee a_1c_2 = (c_1 \vee a_1)c_2 = b_1c_2 = b_2c_2$$

by H, H, H, P 3_2, T 13 — T 13, H, H, H, T 13 — H, P 3_2, H, H.

* THEOREM 16_2. *If* $ba = a$, *the equations of Theorem* 16_1 *have the solution* ba'.

We have $ba' \vee a = ba' \vee ba = b(a' \vee a) = be = b$ by H, P 3_1, *P 5, *T 3; and $(ba')a = 0$ by *L 6_4.

We have now obtained the principal rules governing the operations of addition and multiplication in a Boolean algebra. Both operations are commutative, by Postulate 1_1 and Theorem 13; both are associative, by Theorems 14 and 15; and each is distributive with respect to the other, by Postulates 3_1 and 3_2 and Theorem 11. The further rules $a \vee a = a$, $aa = a$, $a \vee ab = a$ are given by Postulate 6_1, Postulate 6_2, and Theorem 10 respectively. By virtue of the commutative and associative laws, we can simplify our notation: it is no longer necessary to indicate by the use of parentheses or by the ordering of the base elements the precise structure of an additive or multiplicative construct or polynomial in a Boolean algebra. In Definition 2 we have introduced the new unary operation of complementation; the principal rules which govern it are embodied in the relations $a'' = a$, $a' \vee a = e$, $a'a = 0$, $(a \vee b)' = a'b'$, and $(ab)' = a' \vee b'$ of Theorem 8, Definition 2, Definition 2, Theorem 12_1, and Theorem 12_2, respectively. The last two relations reveal clearly the complete duality between the operations of addition and multiplication, which one would suspect from a comparison of the rules they obey: it is evident that any rule connecting addition and multiplication can be converted, by taking complements, into a rule in which the rôles of the two operations are interchanged. A systematic treatment of this duality, while unnecessary for our purposes, can easily be supplied. Finally, we have presented in Postulate 5 and Theorems 16_1 and 16_2 those facts from the theory of equations which will be most useful in the sequel. Since any algebra of classes satisfies the postulates of Definition 1 or is a subsystem of an algebra with this property, we can apply these results to the operations upon classes; it is easily seen that the operation of complementation here becomes the familiar operation of forming the complement of a given class.

We shall now study the definition of a relation in a Boolean algebra which is analogous to the relation $\subset$ defined for classes. We can introduce such a relation in any one of a number of equivalent ways. Here we choose to make our fundamental definition in terms of the operation of multiplication.

DEFINITION 3. *If* $ab = a$, *we write* $a < b$, $b > a$; *and we say that* a *is less than* b, b *greater than* a.

THEOREM 17_1. $0 < a$ *for every element* a.

We have $0a = 0$ for every a by T 6.

* THEOREM 17_2. *$a < e$ for every element a.*

We have $ae = a$ for every a by *T 3.

THEOREM 18_1. *If $a < b$ and $b < c$, then $a < c$.*

We have $ac = (ab)c = a(bc) = ab = a$ by H, T 15, H, H.

THEOREM 18_2. *If $a < b$, $b < c$, and $a = c$, then $a = b = c$.*

We have $c = a = ab = cb = bc = b$ by H, H, H, T 13, H.

THEOREM 18_3. *If $a < b$ and $b < a$, then $a = b$; and conversely.*

If $a < b$ and $b < a$, we have $a = a$ and hence $a = b$ by T 18_2. If $a = b$, we have $ab = aa = a$ by P 6_2; and $ba = bb = b$ by P 6_2.

THEOREM 19. *If $a < b$ and $c < d$, then $ac < bd$ and $a \vee c < b \vee d$.*

We have $(ac)(bd) = (ab)(cd) = ac$ by T 13 — T 15, H; and $(a \vee c)(b \vee d) = (ab \vee ad) \vee (cb \vee cd) = (a \vee ad) \vee (cb \vee c) = a \vee c$ by P 3_1 — P 3_2, H, T 10 — P 1_1 — T 10.

THEOREM 20_1. *In order that $a < b$ and $a < c$, it is necessary and sufficient that $a < bc$.*

In order to prove the sufficiency of the condition $a < bc$, it is sufficient by T 18_1 to show that $bc < b$ and $bc < c$. Now we have $(bc)b = b(bc) = (bb)c = bc$ by T 13, T 15, P 6_2; and $(bc)c = b(cc) = bc$ by T 15, P 6_2. In order to prove the necesssity of the condition $a < bc$, we have merely to note that $a(bc) = (aa)(bc) = (ab)(ac) = aa = a$ by P 6_2, T 13 — T 15, H — H, P 6_2.

THEOREM 20_2. *In order that $a > b$ and $a > c$, it is necessary and sufficient that $a > b \vee c$.*

In order to prove the sufficiency of the condition $a > b \vee c$, it is sufficient by T 18_1 to show that $b \vee c > b$ and $b \vee c > c$. Now we have $b > b$ by T 18_3, $c > 0$ by T 17_1, $b \vee c > b \vee 0$ by T 19, $b \vee 0 > b$ by P 4_1 — T 18_3, and $b \vee c > b$ by T 18_1; and $b \vee c > c \vee b$ by P 1_1 — T 18_3, $c \vee b > c$ by what has just been proved, and $b \vee c > c$ by T 18_1. In order to prove the necessity of the condition $a > b \vee c$, we note that $a \vee a > b \vee c$ by H — H — T 19, $a > a \vee a$ by P 6_1 — T 18_3, and $a > b \vee c$ by T 18_1.

THEOREM 21. *In order that $a < b$, it is necessary and sufficient that $a \vee b = b$.*

If $ab = a$, we have $a \vee b = ab \vee b = b \vee ab = b$ by H, P 1_1, T 10; and, if $a \vee b = b$, we have $ab = a(a \vee b) = aa \vee ab = a \vee ab = a$ by H, P 3_1, P 6_2, T 10.

* THEOREM 22. *In order that $a < b$, it is necessary and sufficient that $a' > b'$.*

If $a < b$, we have $b'a' = a'b' = (a \vee b)' = b'$ by T 13, *T 12_1, T 21; and, if $a' > b'$, we have $ab = ba = (b' \vee a')' = (a')' = a'' = a$ by T 13, *T 12_2, T 21, *D 2, *T 8.

* THEOREM 23_1. *In order that $a < b$, it is necessary and sufficient that $ab' = 0$.*

If $ab = a$, we have $ab' = (ab)b' = 0$ by H, *L 2 — *L 6_4; and if $ab' = 0$, we have $ab = ab \vee 0 = ab \vee ab' = a(b \vee b') = ae = a$ by P 4_1, H, P 3_1, P 1_1 — *P 5, *T 3.

* THEOREM 23_2. *In order that $a < b$, it is necessary and sufficient that $a' \vee b = e$.*

If $a < b$, then $a' \vee b = a' \vee b'' = (ab')' = 0' = e$ by *T 8, *T 12_2, *T 23_1, *T 8; and, if $a' \vee b = e$, then $ab' = (a' \vee b'')' = (a' \vee b)' = e' = 0$ by *T 12_2, *T 8, H, *T 8, so that $a < b$ by *T 23_1.

We have now examined in sufficient detail the relation introduced in Definition 3. We have not distinguished the case where we have $a < b$ and $a \neq b$ from the case where we have merely $a < b$; in the present paper it is not important that we do so. We have not proved, nor can we do so in general, that one of the two relations $a < b$, $b < a$ holds whatever the elements a and b. Thus, in view of Theorem 18_1, we may describe the relation indicated by the symbol $<$ as an incomplete ordering relation. The most striking and important connections between this relation and the operations of addition, multiplication, and complementation are those presented in Theorems 19 and 22.

We next study the introduction [6] of an operation analogous to the operation of forming the symmetric difference of two classes—that is, the class of elements belonging to either one but not both of the given classes. In order to abbreviate the somewhat tedious proofs of several of the theorems which we have to establish—in particular, those of Theorems 32, 42, and 43—we shall

[6] See Daniell, *Bulletin of the American Mathematical Society*, vol. 23 (1916), pp. 446-458.

occasionally be less precise than hitherto in citing earlier propositions. We commence with a variation upon Theorem 16_2.

THEOREM 24. *The equations* $x \vee ab = a \vee b$, $x(ab) = 0$ *have a unique solution.*

If these equations have a solution, it is unique, by Theorem 16_1. We have $(a \vee b)(ab) = (aa)b \vee a(bb) = ab \vee ab = ab$ by P 3_2 — T 13 — T 15, P 6_2, P 6_1. Hence we see that the equations in question have a solution, by *T 16_2.

DEFINITION 4. *The unique solution of the simultaneous equations* $x \vee ab = a \vee b$, $x(ab) = 0$ *is called the symmetric difference of the elements a and b, and is denoted by* $a \Delta b$. *The operation of forming the symmetric difference is called differentiation.*[7]

THEOREM 25. *If* $a = c$ *and* $b = d$, *then* $a \Delta b = c \Delta d$.

By hypothesis, we have $ab = cd$ and $a \vee b = c \vee d$. From T 24 and D 4, we conclude that $a \Delta b = c \Delta d$.

THEOREM 26. $b \Delta a = a \Delta b$.

We have $a \vee b = b \vee a$ by P 1_1 and $ab = ba$ by T 13. Hence we have $b \Delta a = a \Delta b$ by T 16_1 and D 4.

THEOREM 27. $a \Delta b = (a \vee b) \Delta (ab)$.

We have $(a \vee b) \vee ab = (a \vee ab) \vee b = a \vee b$ by P 1_1 — T 14, T 10; and $(a \vee b)(ab) = ab$ by the proof of T 24. Hence we conclude that $a \Delta b = (a \vee b) \Delta (ab)$ by T 16_1 and D 4.

THEOREM 28. *In order that* $a \Delta b = 0$, *it is necessary and sufficient that* $a = b$; *in particular,* $a \Delta a = 0$ *for every element a.*

If $a \Delta b = 0$, we have $a \vee b = (a \Delta b) \vee ab = 0 \vee ab = ab$ by D 4, H, P 1_1 — P 4_1; we have $ab < a$, $a < a \vee b$ by T 20_1, T 20_2 and $ab < b$, $b < a \vee b$ by T 20_1, T 20_2; hence we conclude that $a = a \vee b = b$ by T 18_2. If $a = b$, we have $a \vee b = a \vee a = a$ by P 6_1 and $ab = aa = a$ by P 6_2; hence $a \Delta b$ is the unique solution of the equations $x \vee a = a$, $xa = 0$ in accordance with T 24 and D 4; since these equations have the solution 0 by P 1_1, P 4_1, T 6, we conclude that $a \Delta b = 0$. Since $a = a$, we have in particular $a \Delta a = 0$.

[7] This term is introduced on the basis of its colloquial logical connotations. There is little danger that it will be confused with the term "differentiation" used in mathematical analysis with a special technical meaning.

THEOREM 29. $a \Delta 0 = a$.

We have $a \vee 0 = a$ by P 4_1 and $a0 = 0$ by T 6; hence $a \Delta 0$ is the unique solution of the equations $x \vee 0 = a$, $x0 = 0$ in accordance with T 24 and D 4; since these equations have the solution a by P 4_1, T 6, we see that $a \Delta 0 = a$.

* THEOREM 30. $a \Delta e = a'$.

We have $a \vee e = e$ by *T 5 and $ae = a$ by *T 3; hence $a \Delta e$ is the unique solution of the equations $x \vee a = e$, $xa = 0$ in accordance with T 24 and D 4; since these equations have the solution a' by *P 5 and *D 2, we see that $a \Delta e = a'$.

* THEOREM 31. $a \Delta b = ab' \vee a'b$.

We have $a \Delta b = (a \vee b)(ab)' = (a \vee b)(a' \vee b') = aa' \vee ab' \vee a'b \vee bb'$ $= 0 \vee ab' \vee a'b \vee 0 = ab' \vee a'b$ by *T 16_2, *T 12_1, P 3_1 — P 3_2 — T 13 — T 14, *L 3, P 1_1 — T 14 — P 4_1.

THEOREM 32. $(a \Delta b) \Delta (b \Delta c) = a \Delta c$.

Using *T 31, the commutative and associative laws for addition and multiplication, and the familiar properties of complementation, we have

$$\begin{aligned}(a \Delta b) \Delta (b \Delta c) &= (ab' \vee a'b)(bc' \vee b'c)' \vee (ab' \vee a'b)'(bc' \vee b'c) \\ &= (ab' \vee a'b)(b' \vee c)(b \vee c') \vee (a' \vee b)(a \vee b')(bc' \vee b'c) \\ &= (ab' \vee a'b)(b'c' \vee bc) \vee (a'b' \vee ab)(bc' \vee b'c) = ab'c' \vee a'bc \vee a'b'c \vee abc' \\ &= ac'(b' \vee b) \vee a'c(b \vee b') = ac'e \vee a'ce = ac' \vee a'c = a \Delta c.\end{aligned}$$

THEOREM 33. *If $c = a \Delta b$, then $a = b \Delta c$; in particular $a = b \Delta (a \Delta b)$.*

If $c = a \Delta b$, we have

$$a = a \Delta 0 = (a \Delta b) \Delta (b \Delta 0) = (a \Delta b) \Delta b = b \Delta (a \Delta b) = b \Delta c$$

by T 29, T 32, T 29 — T 25, T 26, H — T 25.

THEOREM 34. $a \Delta (b \Delta c) = (a \Delta b) \Delta c$.

We have $a \Delta (b \Delta c) = [a \Delta b] \Delta [b \Delta (b \Delta c)] = (a \Delta b) \Delta c$ by T 32, T 26 — T 33 — T 25.

THEOREM 35. *In order that $a \Delta b = a \Delta c$ it is necessary and sufficient that $b = c$.*

By T 28, $a \Delta b = a \Delta c$ if and only if $(a \Delta b) \Delta (a \Delta c) = 0$; the latter relation holds, by T 26, T 25, T 32, if and only if $b \Delta c = 0$; and, by T 28, $b \Delta c = 0$ if and only if $b = c$.

THEOREM 36. $ab = (a \vee b) \Delta (a \Delta b)$.

This result follows at once from T 27, T 26, T 33.

THEOREM 37. $a \vee b = (ab) \Delta (a \Delta b)$.

This result follows at once from T 27, T 26, T 33.

THEOREM 38. $(a \Delta b)c = (ac) \Delta (bc)$.

We have

$$(ac) \Delta (bc) = (ac)(bc)' \vee (ac)'(bc) = (ac)(b' \vee c') \vee (a' \vee c')(bc)$$
$$= (ab')c \vee (a'b)c = (ab' \vee a'b)c = (a \Delta b)c$$

by T 31, and the familiar properties of addition, multiplication, and complementation.

THEOREM 39. $a \vee (a \Delta b) = a \vee b$.

We have

$$a \vee (a \Delta b) = (a \vee ab) \vee (a \Delta b) = a \vee [(a \Delta b) \vee ab] = a \vee (a \vee b) = (a \vee a) \vee b = a \vee b$$

by T 10, P 1_1 — T 14, D 4, T 14, P 6_1.

* THEOREM 40. $(a \Delta b)' = a' \Delta b = a \Delta b'$.

We have $(a \Delta b)' = (ab' \vee a'b)' = (a' \vee b)(a \vee b') = a'b' \vee ab = a' \Delta b$ by *T 31, and the familiar properties of addition, multiplication and complementation. By T 26, we have also $(a \Delta b)' = (b \Delta a)' = b' \Delta a = a \Delta b'$. We can also obtain this result from T 30, T 34, and T 26, in an obvious manner.

THEOREM 41. *If* $a < c$ *and* $b < d$, *then* $a \Delta b < c \vee d$; *in particular,* $a \Delta b < a \vee b$.

By T 18_1 and T 19, it is sufficient to show that $a \Delta b < a \vee b$. Now we have $(a \Delta b) \vee ab = a \vee b$ by D 4; and hence $(a \Delta b) < a \vee b$ by T 18_3 and T 20_2.

THEOREM 42. $a \Delta c < (a \Delta b) \vee (b \Delta c)$.

This result follows directly from T 32 by T 18_1, T 18_3, and T 41.

THEOREM 43. $(ab) \Delta (cd) < (a \Delta c) \vee (b \Delta d)$.

By suitable applications of *T 31 and the familiar properties of addition, multiplication, and complementation, we have

$$(ab) \Delta (cd) = (ab)(cd)' \vee (ab)'(cd) = (ac')b \vee (bd')a \vee (a'c)d \vee (b'd)c$$

and

$$(a \Delta c) \vee (b \Delta d) = ac' \vee bd' \vee a'c \vee b'd.$$

Then by using T 19 together with the relations $(ac')b < ac'$, $(bd')a < bd'$, $(a'c)d < a'c$, $(b'd)c < b'd$, which all follow from T 20_1, we obtain the result asserted in the statement of the theorem.

THEOREM 44. $(a \vee b) \Delta (c \vee d) < (a \Delta c) \vee (b \Delta d)$.

The proof is similar to that of the preceding theorem. We have

$$(a \vee b) \Delta (c \vee d) = (a \vee b)(c \vee d)' \vee (a \vee b)'(c \vee d)$$
$$= (ac')d' \vee (bd')c' \vee (a'c)b' \vee (b'd)a' < ac' \vee bd' \vee a'c \vee b'd = (a \Delta c) \vee (b \Delta d).$$

We have now obtained the chief properties of the operation of differentiation. From an algebraic point of view, differentiation is the analogue in a Boolean algebra of both addition and subtraction in an abstract ring.[8] The commutative and associative laws hold for differentiation, as we have shown in Theorems 26 and 34 respectively; and multiplication is distributive with respect to differentiation by Theorem 38. The connections between differentiation and the operations of addition, multiplication, and complementation are given by Theorems 37 and 39, Theorem 36, and * Theorems 31 and 40, respectively. In view of Theorems 25 and 37, we may regard a Boolean algebra as a system with double composition in which the given operations are those which we here call multiplication and differentiation, the operation which we here call addition being introduced through the equation $a \vee b = (ab) \Delta (a \Delta b)$. Alternatively, we may treat addition and differentiation as the given operations and introduce multiplication through the equation $ab = (a \vee b) \Delta (a \Delta b)$ of Theorem 36. We shall next call attention to an analogy which throws a good deal of light on Theorems 26, 28, and 42. If we call $a \Delta b$ the Boolean distance from the element a to the element b, we find that these theorems simply reflect familiar properties of ordinary numerical distances: Theorems 17 and 25 show that the Boolean distance is a function of the arguments a and b defined and single-valued for all a and b in a Boolean algebra A, with

[8] See the remarks in the introduction.

values greater than (or equal to) 0 belonging to A; Theorem 26 shows that the Boolean distance is symmetric in its two arguments; Theorem 28 shows that the Boolean distance is equal to 0 if and only if $a = b$; Theorem 42 shows that the Boolean distance satisfies the "triangle inequality"; and Theorem 38 shows that the Boolean distance is a homogeneous function of degree one under Boolean multiplication of its arguments. Finally, we point out that the reciprocity established in Theorem 33 can be brought out in another way which is interesting in itself but not important later. Let us define a triadic relation by writing $R(a, b, c)$ to indicate that the equations $a(b \vee c) = a$, $b(c \vee a) = b$, $c(a \vee b) = c$, $abc = 0$ hold for the elements a, b, c. It is evident that this relation is symmetric in the related elements—in other words, that the relation persists under any permutation of the elements a, b, c. We can now show that, when b and c are given, there exists a unique element a such that $R(a, b, c)$—namely, the element $a = b \Delta c$: for $R(a, b, c)$ implies $a \vee bc = (b \vee c)a \vee bc = (ba \vee bc) \vee (ac \vee bc) = b \vee c$, $a(bc) = 0$, and hence $a = b \Delta c$; and $a = b \Delta c$ implies

$$a(b \vee c) = (b \Delta c)[bc \Delta (b \Delta c)] = (b \Delta c)bc \Delta (b \Delta c) = 0 \Delta a = a,$$
$$b(c \vee a) = b[c \vee (b \Delta c)] = b(b \vee c) = b,$$
$$c(a \vee b) = c[(b \Delta c) \vee b] = c(c \vee b) = c, \qquad abc = (b \Delta c)bc = 0,$$

and hence $R(a, b, c)$. Thus the symmetry of our triadic relation is reflected in the reciprocity of Theorem 33.

2. *Connections with the Postulates of Huntington and Del Re.* We must now show that our definition of Boolean algebras is essentially equivalent to at least one of the standard definitions. Naturally it is convenient to make the necessary comparison with some definition which, like ours, is based upon a system with double composition. The two best-known sets of postulates of this type are those due respectively to Huntington and to Del Re. We repeat them here:[9]

HUNTINGTON'S POSTULATES. *A Boolean algebra A is a system with double composition which satisfies the postulates*

Postulate 1_1. $a \vee b = b \vee a$; Postulate 1_2. $ab = ba$;

Postulate 3_1. $a(b \vee c) = ab \vee ac$;

Postulate 3_2. $a \vee bc = (a \vee b)(a \vee c)$;

[9] See Huntington, *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 288-309; Del Re, *Algebra della Logica*, Naples, 1907; B. A. Bernstein, *University of California Publications in Mathematics*, vol. 1 (1914), pp. 87-96.

Postulate 4_1. *There exists an element* 0 *in* A *such that* $a \vee 0 = a$ *for every element* a *in* A;

Postulate 4_2. *There exists an element* e *in* A *such that* $ea = a$ *for every element* a *in* A;

Postulate 5_3. *If the elements* 0 *and* e *of Postulates* 4_1 *and* 4_2 *respectively exist and are unique, then the simultaneous equations* $x \vee a = e$, $xa = 0$ *have a solution for every element* a *in* A;

Postulate 7. *There exist elements* a *and* b *in* A *such that* $a \neq b$.

DEL RE'S POSTULATES. *A Boolean algebra* A *is a system with double composition which satisfies the postulates*

Postulates $1_1, 1_2, 3_1, 4_1, 4_2, 5_3, 6_1, 7$; [*Postulate* 2_1. $(a \vee b) \vee c = a \vee (b \vee c)$.]

We have, of course, numbered the various postulates concerned in such a way as to facilitate comparison with our own system. Furthermore, we have made some minor changes in phraseology in order to eliminate trivial differences in statement.

We now observe that, of the postulates which occur in Huntington's and Del Re's systems, every one except Postulate 7 either occurs in our system or is deducible therefrom. We have established Postulate 1_2 as Theorem 13 in our system, Postulate 2_1 as Theorem 14, Postulate 3_3 as Theorem 11, Postulate 4_2 as Theorem 3; and we can establish Postulate 5_3 as a consequence of Postulate 5 and Theorem 3.

We next observe that, of the postulates which occur in our system, every one either occurs in Huntington's system or is deducible therefrom without the use of Postulate 7. We can establish Postulate 3_2 in Huntington's system since $(a \vee b)c = c(a \vee b) = ca \vee cb = ac \vee bc$ by P 1_2, P 3_1, P 1_2 — P 1_2. We can establish Postulate 5 as follows: in Huntington's system we establish the existence and uniqueness of the element 0 by combining Postulates 1_1 and 4_1 just as we did in proving Theorem 1 in our system; we establish the existence and uniqueness of the element e by combining Postulates 1_2 and 4_2 in a similar manner; we then use Postulate 5_3 to show that, for the indicated elements 0 and e, the simultaneous equations $x \vee a = e$, $xa = 0$ have a solution, whatever the element a in A. We establish Postulate 6_1 from the relations $a = a \vee 0 = a \vee xa = (a \vee x)(a \vee a) = (x \vee a)(a \vee a) = e(a \vee a) = a \vee a$ by P 4_1, P 5_3, P 3_3, P 1_1, P 5_3, P 4_2, the element x being chosen as a solution of the simultaneous equations $x \vee a = e$, $xa = 0$, in accordance with Postulate 5_3 and the previous remarks concerning the existence and uniqueness of the elements 0 and e. Finally, we establish Postulate 6_2 from the relations

$a = ea = (x \vee a)a = xa \vee aa = 0 \vee aa = aa \vee 0 = aa$ by P 4_2, P 5_3, P 3_1, P 5_3, P 1_1, P 4_1, the element x being chosen as in the preceding demonstration.

We observe finally that, of the postulates which occur in our system, every one either occurs in Del Re's system or is deducible therefrom without the use of Postulates 2_1 and 7. In fact, we see that Postulates 3_2, 5, and 6_2 can be established in Del Re's system by arguments precisely the same as those which were effective in Huntington's. *Since Postulate* 2_1 *can now be deduced as Theorem* 14 *from the postulates of our system, it follows that this postulate is redundant in Del Re's system.*[10]

Postulate 7 plays no part in deductions from the systems of Huntington and Del Re; indeed, it serves only to exclude the essentially trivial case of an algebra with exactly one element. It is largely a matter of taste whether one shall or shall not reject this case. We find it convenient to consider one-element algebras and hence to omit Postulate 7 from our list.

We may summarize the foregoing comments as follows:

THEOREM 45. *Definition* 1 *is equivalent to Huntington's definition of Boolean algebras and also to Del Re's, provided that Postulate* 7 *is suppressed from their lists of postulates:*

3. *Postulates for generalized Boolean algebras.* It is of some interest to examine the possibility of introducing a system with double composition which possesses most of the peculiar properties of Boolean algebras without containing a unit element. The analogy with the theory of abstract rings suggests that the direct or indirect postulation of the unit element should be avoided; and examples from the theory of classes—for instance, the example of all finite subclasses of a given infinite class, or the example of all Lebesgue or Borel measurable sets of finite measure in n-space—indicate the existence of interesting algebras without unit. We shall therefore introduce the following definition:

DEFINITION 5. *A generalized Boolean algebra A is a system with double composition which satisfies the further postulates:*

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[10] The example which has been given by B. A. Bernstein, *University of California Publications in Mathematics*, vol. 1 (1914), pp. 87-96, especially p. 95, to show the independence of Postulate 2_1 in Del Re's system is known to be irrelevant, as Professor Bernstein informed the writer, first by word of mouth and then by letter when consulted on the situation presented here: for both Postulate 2_1 and Postulate 3_1 fail to hold in this example, as we see from the relations

$$(e_1 \vee e_2) \vee e_2 = e_{12} \vee e_2 = e_2 \neq e_{123} = e_1 \vee e_{23} = e_1 \vee (e_2 \vee e_3)$$

and

$$e_2(e_{12} \vee e_3) = e_2 e_2 = e_0 \neq e_2 = e_2 \vee e_0 = e_2 e_{12} \vee e_2 e_3.$$

Postulate 1_1. $a \vee b = b \vee a$;

Postulate 2_2. $a(bc) = (ab)c$;

Postulate 3_1. $a(b \vee c) = ab \vee ac$;

Postulate 4_1. *There exists an element* 0 *in* A *such that* $a \vee 0 = a$ *for every element* a *in* A;

Postulate 5_1. *If there exists an element* 0 *with the property required by Postulate* 4_1 *and if* a *and* b *are elements of* A *such that* $ba = a$, *then there exists at least one such element* 0, *independent of* a *and* b, *for which the simultaneous equations* $x \vee a = b$, $xa = 0$ *have a solution in* A;

Postulate 5_2. *If there exists an element* 0 *with the property required by Postulate* 4_1 *and if* a *and* b *are elements of* A *such that* $ab = a$, *then there exists at least one such element* 0, *independent of* a *and* b, *for which the simultaneous equations* $x \vee a = b$, $ax = 0$ *have a solution in* A;

Postulate 6_1. $a \vee a = a$;

Postulate 6_2. $aa = a$.

Any element such as that postulated in Postulate 4_1 *is called a zero.*

The list of postulates given in this definition is obtained by certain obvious modifications of the list given in Definition 1. We begin by replacing * Postulate 5 by the more complicated, symmetrically related Postulates 5_1 and 5_2, in order to obtain a system without postulated unit in which the indicated simultaneous equations still possess solutions; we may point out that, in the presence of the remaining postulates of Definition 1, the relations $ba = a$ and $ab = a$ of the new postulates are necessary conditions for the existence of such solutions. This substitution unfortunately renders the list of postulates inadequate for our purpose and therefore forces us to proceed further. We elect to add Postulate 2_2 at this stage. We find, however, that the addition makes each of the Postulates 3_1, 3_2 redundant in the presence of the others [11]; but we are able to bring our series of changes to a close with the suppression of one or the other of these two postulates. We have chosen to postulate the distributive law for multiplication on the left; but, if we had chosen to postulate the distributive law for multiplication on the right, we could have recovered the list of postulates given in Definition 5 simply by introducing a new operation o defined by the relation $a \circ b = ba$. It may be noted that the

[11] I am indebted to Professor A. A. Bennett for this observation; the argument given in Theorems 47-50 below is due essentially to him.

finite subclasses of a fixed infinite class constitute a system satisfying the postulates of Definition 5 but possessing no unit.

By way of justifying the terminology introduced in Definition 5, we may note the following result:

THEOREM 46. *Every Boolean algebra is a generalized Boolean algebra.*

We have only to prove that any system with double composition which satisfies the postulates of Definition 1 satisfies the additional postulates occurring in Definition 5. Now Postulate 2_2 has been established as Theorem 15, Postulate 5_1 has been established as Theorem 16_2, and Postulate 5_2 follows from Postulate 5_1 by an application of Theorem 13.

In discussing generalized Boolean algebras, it is our aim to show that all the unstarred theorems of § 1 remain valid in these more general systems. We shall condense the necessary developments as much as possible, using the form of presentation adopted in § 1.

THEOREM 47. *The element* 0 *of Postulate* 4_1 *exists and is unique.*

By P 4_1, such an element exists. If 0_1 and 0_2 are such elements, then $0_1 = 0_1 \vee 0_2 = 0_2 \vee 0_1 = 0_2$ by P 4_1, P 1_1, P 4_1.

We shall use Theorem 47 without specific reference in the sequel.

THEOREM 48_1. $0a = 0$.

By P 6_2 — P 5_1, there exists an element x with the property X that $x \vee a = a$, $xa = 0$. We now have $x = x \vee 0 = xx \vee xa = x(x \vee a) = xa = 0$ by P 4_1, P 6_2 — X, P 3_1, X, X. Combining this result with X, we have $0a = xa = 0$.

THEOREM 48_2. $a0 = 0$.

By P 6_2 — P 5_2, there exists an element x with the property X that $x \vee a = a$, $ax = 0$. Hence we have $a0 = a(ax) = (aa)x = ax = 0$ by X, P 2_2, P 6_2, X.

LEMMA 11. $ab = 0$ *implies* $ba = 0$.

We have $ba = (ba)(ba) = b[a(ba)] = b[(ab)a] = b[0a] = b0 = 0$ by P 6_2, P 2_2, P 2_2, H, T 48_1, T 48_2.

LEMMA 12. $(ab)a = ab$.

We have $ab = (aa)b = a(ab)$ by P 6_2, P 2_2. Hence by P 5_1 there exists an element x with the property X_1 that $x \vee ab = a$, $x(ab) = 0$; and, by

X_1 — L 11, this element x has also the property X_2 that $(ab)x = 0$. Thus we have $(ab)a = (ab)(x \vee ab) = (ab)x \vee (ab)(ab) = 0 \vee ab = ab \vee 0 = ab$ by X_1, P 3_1, X_2 — P 6_2, P 1_1, P 4_1.

THEOREM 49. $ab = ba$.

We have $(ab)(ba) = [(ab)b]a = [a(bb)]a = [ab]a = ab$ by P 2_2, P 2_2, P 6_2, L 12. Hence by P 5_2 there exists an element x with the property X_1 that $x \vee ab = ba$, $(ab)x = 0$. By X_1 — L 11 this element has the property X_2 that $x(ab) = 0$. It has also the property X_3 that $x(ba) = 0$, since

$$(ba)x = [(ba)b]x = b[(ab)x] = b0 = 0$$

by L 12, P 2_2 — P 2_2, X_1, T 48_2, and L 11 is therefore applicable. Finally, it has the property X_4 that $x = 0$, since

$$x = x \vee 0 = xx \vee x(ab) = x(x \vee ab) = x(ba) = 0$$

by P 4_1, P 6_2 — X_2, P 3_1, X_1, X_3. Hence we conclude that

$$ab = ab \vee 0 = ab \vee x = x \vee ab = ba$$

by P 4_1, X_4, P 1_1, X_1.

THEOREM 50. $(a \vee b)c = ac \vee bc$.

We have $(a \vee b)c = c(a \vee b) = ca \vee cb = ac \vee bc$ by T 49, P 3_1, T 49 — T 49; in other words, Postulate 3_2 is valid in a generalized Boolean algebra.

If a is an arbitrary element of A, we may consider the class of all elements c such that $c = ab$ for some element b in A. We shall denote this class by the symbol $\mathfrak{a}(a)$.

THEOREM 51. *If a is an arbitrary element in the generalized Boolean algebra A, then the class $\mathfrak{a}(a)$ is a subsystem of A and is in addition a Boolean algebra, its zero being the element 0 in A and its unit being the element a.*

If c_1 and c_2 are elements of $\mathfrak{a}(a)$, we can write $c_1 = ab_1$, $c_2 = ab_2$; thus, $c_1 \vee c_2 = ab_1 \vee ab_2 = a(b_1 \vee b_2)$ by P 3_1 and $c_1c_2 = (ab_1)(ab_2) = a[b_1(ab_2)]$ by P 2_2; and hence $c_1 \vee c_2$ and c_1c_2 are elements of $\mathfrak{a}(a)$. Thus we see that $\mathfrak{a}(a)$ is a subsystem of the system A. We now note that 0 and a are both elements of $\mathfrak{a}(a)$, since we have $0 = a0$ by T 48_2 and $a = aa$ by P 6_2. Now, since $\mathfrak{a}(a)$ is a subsystem of A containing the element 0 and since Postulates 1_1, 3_1, 3_2 (by T 50), 4_1, 6_1, and 6_2 are valid in A, these postulates are also valid in

$\mathfrak{a}(a)$. Hence, to prove that $\mathfrak{a}(a)$ is a Boolean algebra, we have only to establish Postulate 5 for this system. This may be done as follows: we first note that if c is any element in $\mathfrak{a}(a)$ we can write $c = ab$ and therefore have $ac = a(ab) = (aa)b = ab = c$ by H, P 2_2, P 6_2, H; hence, by P 5_1, there exists an element x in A with the property X that $x \vee c = a$, $xc = 0$; and, since $ax = xa = x(x \vee c) = xx \vee xc = x \vee 0 = x$ by T 49, X, P 3_1, P 6_2 — X, P 4_1, this element x, as well as the elements 0, a, and c, belongs to $\mathfrak{a}(a)$. In particular, we see from the foregoing results that the zero in $\mathfrak{a}(a)$ is the element 0 in A and that the unit in $\mathfrak{a}(a)$ is the element a.

THEOREM 52. *If c is an element of $\mathfrak{a}(a)$, then $\mathfrak{a}(c) \subset \mathfrak{a}(a)$.*

If d is an element in $\mathfrak{a}(c)$, we can write $d = cb_1$ and $c = ab_2$ and thus find that $d = cb_1 = (ab_1)b_2 = a(b_1b_2)$ by H, H, P 2_2. Hence d is in $\mathfrak{a}(a)$ and $\mathfrak{a}(c) \subset \mathfrak{a}(a)$.

THEOREM 53. *The class $\mathfrak{a}(a \vee b)$ contains the elements a and b.*

Since a and ab are elements of the Boolean algebra $\mathfrak{a}(a)$ in accordance with T 51, we have $a \vee ab = a$ by T 10 and hence

$$a = a \vee ab = aa \vee ab = a(a \vee b) = (a \vee b)a$$

by P 6_2, P 3_1, T 49; it follows that a is an element of $\mathfrak{a}(a \vee b)$. Since b and ba are elements of the Boolean algebra $\mathfrak{a}(b)$, we have similarly

$$b = b \vee ba = ba \vee b = ba \vee bb = b(a \vee b) = (a \vee b)b$$

by T 10, P 1_1, P 6_2, P 3_1, T 49 and hence conclude that b is an element of $\mathfrak{a}(a \vee b)$.

THEOREM 54. *The class $\mathfrak{a}(a \vee [b \vee (c \vee d)])$ contains the elements a, b, c, d.*

By T 53 we see that $\mathfrak{a}(a \vee [b \vee (c \vee d)])$ contains a and $b \vee (c \vee d)$, $\mathfrak{a}(b \vee (c \vee d))$ contains b and $c \vee d$, and $\mathfrak{a}(c \vee d)$ contains c and d. By T 52 we see that $\mathfrak{a}(c \vee d) \subset \mathfrak{a}(b \vee (c \vee d)) \subset \mathfrak{a}(a \vee [b \vee (c \vee d)])$. Hence the class $\mathfrak{a}(a \vee [b \vee (c \vee d)])$ contains the elements a, b, c, d.

THEOREM 55. *All unstarred propositions and definitions of § 1 are valid in a generalized Boolean algebra, provided that the term "Boolean algebra" is supplanted wherever it occurs by the term "generalized Boolean algebra."*

A brief survey of the definitions and propositions of § 1 shows that the starred definitions and propositions are precisely those involving the unit element e and the related operation of complementation. Such definitions and theorems obviously cannot be carried over to a generalized Boolean algebra.

To assert that the remaining definitions and propositions of § 1 can be carried over to a generalized Boolean algebra therefore means that the operations of addition, multiplication, and differentiation, and the relation "less than" are defined in such an algebra and are subject to the same formal rules as in a Boolean algebra.

In verifying this assertion, we must examine in succession the various unstarred definitions and propositions of § 1. In deciding whether or not a definition or proposition remains valid in a generalized Boolean algebra, we may appeal to several different criteria. A definition can be retained whenever it is stated directly in terms of the fundamental postulated operations of addition and/or multiplication or indirectly through preceding unstarred definitions and/or propositions. This criterion applies at once to Definitions 3, 4; of course, Definition 4 reposes upon the preceding unstarred Theorem 24. A proposition which is stated in terms of the fundamental postulated operations and/or preceding unstarred definitions remains valid (1) whenever it is one of those postulated or proved in the present section; (2) whenever it reposes only upon preceding unstarred propositions and/or starred propositions which have adequate substitutes valid in a generalized Boolean algebra; (3) whenever it asserts a relation which can be treated by regarding the related elements as belonging to a Boolean algebra constructed in accordance with Theorems 51 and 54 of the present section. We apply criterion (1) in the following instances: Postulates 1_1, 3_1, 4_1, 6_1, 6_2 of Definition 1 appear as postulates in Definition 5; Postulate 3_2 has been proved as Theorem 50 of the present section; Theorem 1 appears essentially as Theorem 47 of the present section; Theorem 6 appears, subdivided as Theorems 48_1 and 48_2 of the present section; Theorem 13 appears as Theorem 49 of the present section and Theorem 15 appears as Postulate 2_2 of Definition 5. We can apply criterion (2) above in the following instances: Theorems 16_1, 17, 18_1, 18_2, 19, 20_1, 20_2, 21, 24, 25, 26, 27, 28, 29, 33, 34, 35, 36, 37, 39, 41, 42, and Lemmas 8_1, 8_2, 10. In the case of Theorem 24, the proof involves the starred Theorem 16_2, which is used, however, merely to establish the existence of a solution of the simultaneous equations under consideration; but in a generalized Boolean algebra the existence of the desired solution is established by Postulates 5_1 and 5_2 (which we may now regard as equivalent by virtue of the commutative law for multiplication). Finally we apply criterion (3) in the following instances: Theorems 9, 10, 11, 14, 32, 43, and 44. We shall content ourselves with giving detailed discussions of two typical cases. First let us consider the demonstration of Theorem 9: if 0 is the zero element of a generalized Boolean algebra A and a and b are arbitrary elements in A with the

property that $a \vee b = 0$, we may regard 0, a, b as elements of the Boolean algebra $\mathfrak{a}(a \vee [b \vee (c \vee d)])$ in accordance with Theorems 51 and 54, the element 0 being the zero of this algebra; Theorem 9 is now applicable within this Boolean algebra and shows that $a = b = 0$, as desired. Let us also consider the demonstration of Theorem 32: whether we regard the elements a, b, c and the operation Δ as belonging to the generalized Boolean algebra A or to the Boolean algebra $\mathfrak{a}(a \vee [b \vee (c \vee d)])$ contained in A, the elements $a \Delta b$, $b \Delta c$, $a \Delta c$ are obtained as the unique solutions of certain fixed pairs of simultaneous equations and are therefore independent of the interpretation chosen; now under the second interpretation, and hence also under the first, these elements belong to the algebra $\mathfrak{a}(a \vee [b \vee (c \vee d)])$; in the same way we find that the element $(a \Delta b) \Delta (b \Delta c)$ also belongs to this algebra; thus we can apply Theorem 32 within the Boolean algebra $\mathfrak{a}(a \vee [b \vee (c \vee d)])$ and conclude that $(a \Delta b) \Delta (b \Delta c) = a \Delta c$, as we wished to do. In summary, we may say that the three indicated criteria cover all the unstarred propositions of § 1 and thus lead to the present theorem.

We shall conclude the present section with a brief discussion of certain conditions under which a generalized Boolean algebra is a Boolean algebra. We have first:

THEOREM 56. *A necessary and sufficient condition that a generalized Boolean algebra A be a Boolean algebra is that it contain an element e with one or both of the equivalent properties that $ae = a$, $ea = a$ whatever the element a in A. If such an element exists, it is the unit of the Boolean algebra A.*

If A is a Boolean algebra, such an element exists by Postulates 4_1, 5 and Theorems 2, 3. If such an element exists, then Postulates 5_1, 5_2 show that the simultaneous equations $x \vee a = e$, $xa = 0$ have a solution whatever the element a in A; thus the system A satisfies Postulate 5 as well as all the unstarred postulates of Definition 1, and is therefore a Boolean algebra with the element e as its unit.

THEOREM 57. *If A is a generalized Boolean algebra with finite cardinal number n, then A is a Boolean algebra.*

By hypothesis, there exist elements $a_1, \cdots, a_n$ such that $a_j \neq a_k$ for $j \neq k$, $j, k = 1, \cdots, n$ while $a \in A$ implies $a = a_k$ for some integer k, $k = 1, \cdots, n$. Now the element $e = a_1 \vee \cdots \vee a_n$ has the property that $ae = a$ whatever the element a in A: for, putting $a = a_k$, we have

$$ae = a_k(a_1 \vee \cdots \vee a_n) = a_k a_k \vee a_k (a_1 \vee \cdots \vee a_{k-1} \vee a_{k+1} \vee \cdots \vee a_n) = a_k = a$$

by H, P 1_1 — P 1.3_1 — T 14 — T 55, P 6_2 — T 10 — T 55, H. Thus A is a Boolean algebra in accordance with Theorem 56.

4. *Independence of the postulates.* The consistency of the sets of postulates given in Definitions 1 and 5 for Boolean algebras and generalized Boolean algebras respectively is shown by exhibiting specific systems with double composition in which all the listed postulates are valid. The system with a single element 0 where $0 \vee 0$ and 00 are both 0 is one such system; and the system with two elements 0 and e, where $0 \neq e$, where $0 \vee 0$, 00, $0e$, $e0$ are all 0, and where $0 \vee e$, $e \vee 0$, $e \vee e$, ee are all e, is another. As we have already pointed out by means of examples, there are systems with double composition which are generalized Boolean algebras without being Boolean algebras; in view of Theorem 57, such systems are necessarily infinite.

In the following examples, we shall denote by *KPα a system with double composition in which Postulate α of Definition 1 is not satisfied while the remaining postulates of that definition are valid; and we shall denote by KPα a system with double composition in which Postulate α of Definition 5 is not satisfied while the remaining postulates of that definition are valid. In each example, the system considered is finite and its elements are designated by appropriate letters, different letters always denoting unequal elements and the letters 0 and e being reserved for elements with the properties described in Postulates 4_1 and 5; and the fundamental operations are defined by an addition table placed at the left and a multiplication table placed at the right, the tables being arranged so that $a \vee b$ or ab appears in the row labeled "a" and the column labeled "b." In many cases we shall point out various properties of the systems exhibited beyond those mentioned above. In those instances where we assert the failure of a certain proposition in a given system, we shall indicate at least one specific case of the asserted failure; but in those instances where we assert the validity of a certain proposition in a given system, we shall merely report the result of checking all possible cases.

*KP 1_1, KP 1_1.

	0_1	0_2
0_1	0_1	0_1
0_2	0_2	0_2

	0_1	0_2
0_1	0_1	0_1
0_2	0_1	0_2

It is evident that P 1_1 fails here, since $0_1 \vee 0_2 = 0_1 \neq 0_2 = 0_2 \vee 0_1$. P 2_2, P 3_1, P 3_2, P 6_1, P 6_2 are easily verified; and it is also clear that P 4_1 is valid whichever of the two elements 0_1, 0_2 is designated as the element 0 of that

postulate. If we choose 0_1 as the element 0 and also as the element e of *P 5, we find that *P 5 is satisfied: for the simultaneous equations $x \vee a = 0_1$, $xa = 0_1$ have the solution 0_1, whatever the element a.

Neither P 5_1 nor P 5_2 is valid here: for none of the four sets of simultaneous equations

$$x \vee 0_2 = 0_2, \quad x0_2 = 0_1; \qquad x \vee 0_2 = 0_2, \quad 0_2x = 0_1;$$
$$x \vee 0_1 = 0_2, \quad x0_1 = 0_2; \qquad x \vee 0_1 = 0_2, \quad 0_1x = 0_2;$$

has a solution although the relations $0_20_2 = 0_2$, $0_10_2 = 0_20_1 = 0_1$ are valid. It is easily verified that Theorems 10, 11, 13, 14, 15 hold in this system.

*KP 3_1.

	0	a	b	c	e
0	0	a	b	c	e
a	a	a	e	e	e
b	b	e	b	b	e
c	c	e	b	c	e
e	e	e	e	e	e

	0	a	b	c	e
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	c	b
c	0	0	b	c	c
e	0	a	b	c	e

The failure of P 3_1 is exhibited by the relations

$$c(a \vee b) = ce = c \neq b = 0 \vee b = ca \vee cb.$$

It can be verified directly that P 1_1, P 3_2, P 4_1, *P 5, P 6_1, P 6_2 are valid in this system; and also that P 2_2, P 5_2, T 9, T 14, T 15 are valid here. On the other hand, P 5_1 fails since $bc = c$ while the simultaneous equations $x \vee c = b$, $xc = 0$ have no solution; T 10 fails since $c \vee cb = c \vee b = b \neq c$; T 11 fails since $(c \vee a)(c \vee b) = eb = b \neq c = c \vee 0 = c \vee ab$; and T 13 fails since $bc = c \neq b = cb$.

*KP 3_2.

	0	a	e
0	0	a	e
a	a	a	e
e	e	e	e

	0	a	e
0	0	0	0
a	a	a	a
e	0	0	e

The failure of P 3_2 is evident from the relations

$$(e \vee a)0 = e0 = 0 \neq a = 0 \vee a = e0 \vee a0.$$

It can be verified directly that P 1_1, P 3_1, P 4_1, *P 5, P 6_1, P 6_2 are valid in this system; and also that P 2_2, P 5_1, T 9, T 14, T 15 are valid here. On the

other hand, P 5_2 fails since $a0 = a$ while the equations $x \vee a = 0$, $ax = 0$ have no solution; T 10 fails since $0 \vee a0 = 0 \vee a = a \neq 0$; T 11 fails since

$$(a \vee e)(a \vee 0) = ea = 0 \neq a = a \vee 0 = a \vee e0;$$

and T 13 fails since $a0 = a \neq 0 = 0a$.

*KP 4_1, KP 4_1.

	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

	a	b	c
a	a	a	a
b	a	b	c
c	a	c	c

The failure of P 4_1 is evident from the addition table. Since the element 0 described in P 4_1 does not exist, *P 5, P 5_1, and P 5_2 are satisfied vacuously. It can be verified directly that P 1_1, P 2_2, P 3_1, P 3_2, P 6_1, P 6_2 are valid in this system; and also that T 11, T 13, T 14, T 15 are valid here. On the other hand, T 10 fails since $b \vee ba = b \vee a = c \neq b$.

*KP 5.

	0	a
0	0	a
a	a	a

	0	a
0	0	a
a	a	a

The failure of *P 5 is evident from the fact that even the single equation $xa = 0$ has no solution in this system. It is easily verified that P 1_1, P 3_1, P 3_2, P 4_1, P 6_1, P 6_2 are valid here; and that P 2_2, T 9, T 11, T 13, T 14, T 15 are also valid. On the other hand, P 5_1 fails since $0a = a$ while the equations $x \vee a = 0$, $xa = 0$ have no solution; P 5_2 fails since $a0 = a$ while the equations $x \vee a = 0$, $ax = 0$ have no solution; and T 10 fails since $0 \vee 0a = 0 \vee a = a \neq 0$.

*KP 6_1, KP 6_1.

	0	e
0	0	e
e	e	0

	0	e
0	0	0
e	0	e

The failure of P 6_1 is evident from the relation $e \vee e = 0$. It is easily verified that P 1_1, P 2_2, P 3_1, P 3_2, P 4_1, *P 5, P 5_1, P 5_2, P 6_2 are valid in this system; and also that T 13, T 14, T 15 are valid here. On the other hand, T 9 fails since $e \vee e = 0$; T 10 fails since $e \vee ee = e \vee e = 0 \neq e$; and T 11 fails since $(e \vee e)(e \vee 0) = 0e = 0 \neq e = e \vee 0 = e \vee e0$. This system is

of particular interest because it is a ring, isomorphic to the ring of integers taken modulo 2.

*KP 6_2, KP 6_2.

	0	e
0	0	e
e	e	e

	0	e
0	0	0
e	0	0

The failure of P 6_2 is evident from the relation $ee = 0$. It is easily verified that P 1_1, P 2_2, P 3_1, P 3_2, P 4_1, *P 5, P 5_1, P 5_2, P 6_1 are valid in this system; and also that T 9, T 10, T 13, T 14, T 15 are valid here. On the other hand, T 11 fails since $(e \vee 0)(e \vee 0) = ee = 0 \neq e = e \vee 0 = e \vee 00$.

KP 2_2.

	0	a	b	c
0	0	a	b	c
a	a	a	c	b
b	b	c	b	a
c	c	b	a	c

	0	a	b	c
0	0	0	0	0
a	0	a	c	b
b	0	c	b	a
c	0	b	a	c

It is evident that P 2_2 fails in this system since

$$a(bc) = aa = a \neq c = cc = (ab)c.$$

It is easily verified that P 1_1, P 3_1, P 4_1, P 5_1, P 5_2, P 6_1, P 6_2 are valid here; and that P 3_2, T 9, T 13 are also valid. On the other hand, *P 5 fails since the relations $x \vee y = z$, $xy = 0$, imply $x = 0$, $z = y$ or $y = 0$, $z = x$ and hence preclude the existence of the required element e; T 10 fails since $a \vee ab = a \vee c = b \neq a$; T 11 fails $(a \vee 0)(a \vee b) = ac = b \neq a = a \vee 0 = a \vee 0b$; T 14 fails since $a \vee (b \vee b) = a \vee b = c \neq a = c \vee b = (a \vee b) \vee b$; and T 15 fails since it is equivalent to P 2_2.

KP 3_1.

	0	a	b	c	e
0	0	a	b	c	e
a	a	a	e	e	e
b	b	e	b	e	e
c	c	e	e	c	e
e	e	e	e	e	e

	0	a	b	c	e
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
e	0	a	b	c	e

The failure of P 3_1 is evident from the relations

$$a(b \vee c) = ae = a \neq 0 = 0 \vee 0 = ab \vee ac.$$

It is easily verified that P 1_1, P 2_2, P 4_1, P 5_1, P 5_2, P 6_1, P 6_2 are valid in this system; and also that *P 5, T 9, T 10, T 13, T 14, T 15 are valid here. On the other hand, T 11 fails since $(a \vee b)(a \vee c) = ee = e \neq a = a \vee 0 = a \vee bc$. This particular system may be interpreted in the following way: e is a fixed two-dimensional linear manifold; 0 is a fixed 0-dimensional linear manifold in e; a, b, c are three distinct 1-dimensional linear manifolds in e containing 0; the sum of any two of these linear manifolds is the least linear manifold containing both, and the product is the largest linear manifold contained in both.

KP 5_1.

	0	a
0	0	a
a	a	a

	0	a
0	0	a
a	0	a

The failure of P 5_1 is evident from the fact that $0a = a$ while the simultaneous equations $x \vee a = 0$, $xa = 0$ have no solution. It is easily verified that P 1_1, P 2_2, P 3_1, P 4_1, P 5_2, P 6_1, P 6_2 are valid in this system; and also that P 3_2, T 9, T 11, T 14, T 15 are valid here. On the other hand, *P 5 fails since neither the equations $x \vee a = 0$, $xa = 0$ nor the equations $x \vee a = a$, $xa = 0$ have a solution and consequently no element with the properties demanded of e exists; T 10 fails since $0 \vee 0a = 0 \vee a = a \neq 0$; and T 13 fails since $0a = a \neq 0 = a0$.

KP 5_2.

	0	a
0	0	a
a	a	a

	0	a
0	0	0
a	a	a

The failure of P 5_2 is evident from the fact that $a0 = a$ while the simultaneous equations $x \vee a = 0$, $ax = 0$ have no solution. It is easily verified that P 1_1, P 2_2, P 3_1, P 4_1, P 5_1, P 6_1, P 6_2 are valid in this system; and also that P 3_2, T 9, T 11, T 14, T 15 are valid here. On the other hand, *P 5 fails since neither the equations $x \vee 0 = a$, $x0 = 0$ nor the equations $x \vee a = 0$, $xa = 0$ have a solution and consequently no element with the properties demanded of e exists; T 10 fails since $0 \vee a0 = 0 \vee a = a \neq 0$; and T 13 fails since $0a = 0 \neq a = a0$. The system under consideration is obtained from KP 5_1 simply by reflecting the multiplication table in its principal diagonal, so that the product xy in KP 5_2 is the same as the product yx in KP 5_1.

In conclusion, we see that each of the various postulates in Definition 1 or Definition 5 is independent of the remaining postulates of that definition.

HARVARD UNIVERSITY.

ON FINITE BOOLEAN ALGEBRAS.[1]

By B. A. BERNSTEIN.

Introduction. In every finite Boolean algebra there exists a set of elements, which I shall call the "minimals" of the algebra, that play a rôle in the algebra very much like that played by the prime numbers in arithmetic. Huntington [2] introduced these elements, under the term "irreducible," [3] for the purpose of proving the order theorem for finite Boolean algebras. The object of my paper is to discuss the minimal elements more fully and to make further application of the properties of these elements. The application will be to some questions concerning sub-algebras of a finite Boolean algebra, to a problem in dichotomy, and to arithmetic representations of finite Boolean algebras. Incidentally, Huntington's results concerning the minimal elements will be obtained from a different starting point. Incidentally, also, a proof of the order theorem for finite Boolean algebras will be obtained without the help of the minimal elements.[4]

1. *The minimals.* Consider any Boolean algebra B. If $a_1, a_2, \cdots, a_n$ be elements of B, and 1 the universe element, then

$$\text{(i)} \qquad 1 = (a_1 + a'_1)(a_2 + a'_2) \cdots (a_n + a'_n)$$

$$\text{(ii)} \qquad = a_1 a_2 \cdots a_n + a_1 a_2 \cdots a'_n + \cdots + a'_1 a'_2 \cdots a'_n.$$

The expression (ii) is the *additive normal development* of 1 with respect to $a_1, a_2, \cdots, a_n$. If B be finite, and if $a_1, a_2, \cdots, a_n$ be *all* the elements of B, (ii) will be called the *complete* additive normal development of 1. The constituents in the complete additive normal development of 1 *that are not* 0 will be called the *minimal elements* of B, or simply the *minimals* of B.

We have at once the following basic theorem:

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[1] Presented to the American Mathematical Society, June 20, 1934.

[2] *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 308, 309.

[3] This term seems to me less simple than "minimal," and permits a less convenient name and notation for the duals of the minimal elements (See § 4 below).

[4] The reader is referred to the following more general papers on Boolean algebra, which no doubt yield, from a very different angle, many of my results: E. T. Bell, "Arithmetic of logic," *Transactions of the American Mathematical Society*, vol. 29 (1927), pp. 597-611; W. A. Hurwitz, "On Bell's arithmetic of Boolean algebra," *ibid.*, vol. 30 (1928), pp. 420-424; M. H. Stone, "On the structure of Boolean algebras" (Abstract), *Bulletin of the American Mathematical Society*, vol. 39 (1933), p. 200; M. H. Stone, "Boolean algebras and their application to topology" (Abstract), *Proceedings of the National Academy of Science*, vol. 20 (1934), pp. 197-202.

1.1. *In a finite Boolean algebra, every element a, $\neq 0$, is expressible uniquely as a sum of minimals.*

For, by (i) and (ii), every element a of the algebra is expressible uniquely as a sum of constituents in (ii), namely, the sum of those constituents in which a enters positively. Hence, if $a \neq 0$, a is expressible uniquely as the sum of those non-0 constituents in which a enters positively.

If m is a minimal in an expression for a, we shall say "m is a minimal of a," or "m belongs to a," or "m is contained in a."

Theorem 1.1 enables us to construct a finite Boolean algebra from its minimals.

Theorems 1.2-1.19 below are practically all simple corollaries of 1.1 and the definition of minimals. Proofs for the more obvious of these theorems will be omitted.

1.2. *Every finite Boolean algebra has at least one minimal.*

1.3. *The order of a finite Boolean algebra is 2^k, where k is the number of minimals of the algebra.*

1.4. *If $m_1, m_2, \cdots, m_k$ are the minimals of a finite Boolean algebra, then*

$$m_1 + m_2 + \cdots + m_k = 1, \qquad m_i m_j = 0 \qquad [i \neq j].$$

1.5. *In a finite Boolean algebra, a necessary and sufficient condition that a be a minimal is:*

(i) $a \neq 0$, (ii) $a + x \neq a$ $[x \neq 0,\ x \neq a]$.

For, first, let a be a minimal. Then $a \neq 0$. Also, if $a + x = a$ $[x \neq 0,\ x \neq a]$, then x, hence $a + x$, hence a would contain minimals not belonging to a, contrary to 1.1. Hence, (i) and (ii) are necessary.

Secondly, let $a \neq 0$ and $a + x \neq a$ $[x \neq 0,\ x \neq a]$. Then, if a were not a minimal, we should have, by 1.1, $a = m_1 + m_2 + \cdots + m_h$ $[h > 1]$, where the elements m_i are minimals. Hence, there would be a minimal, say m_1, such that $a + m_1 = a$, where $m_1 \neq 0$, $m_1 \neq a$, contrary to supposition. Hence, (i) and (ii) are sufficient.

1.6. *In a finite Boolean algebra, a necessary and sufficient condition that a be a minimal is:*

(i) $a \neq 0$, (ii) $x \not< a$ $[x \neq 0,\ x \neq a]$.

For, $x < a$ is equivalent to $a + x = a$. Hence the theorem, by 1.5.

Theorem 1.6 shows that *the minimal elements are identical with Hunting-*

ton's "irreducible" elements. Theorems 1. 1-1. 6 include all of Huntington's results concerning these elements.

1. 7. *Let $a_1, a_2, \cdots, a_k$ be a set of k elements of a finite Boolean algebra B such that*

$$a_1 + a_2 + \cdots + a_k = 1, \qquad a_i \neq 0, \qquad a_i a_j = 0 \qquad [i \neq j];$$

if k be the number of minimals in B, then $a_1, a_2, \cdots, a_k$ are the minimals of B.

For, by 1. 1, each element a_i is a minimal or a sum of minimals; and if some a_i were a sum of two or more minimals, there would be two distinct a_i's having a minimal in common, contrary to the supposition that $a_i a_j = 0$ $[i \neq j]$.

Hereafter, unless expressly stated otherwise, the algebra under consideration in any particular case will be understood to be a finite Boolean algebra, and the symbol m will always denote a minimal.

1. 8. *In the additive normal development of 1 with respect to any set of elements, the number of non-0 constituents cannot exceed the number of minimals.*

For, each non-0 constituent is a sum of minimals, by 1. 1; and no two distinct constituents, c_1 and c_2, can have a common minimal, since $c_1 c_2 = 0$.

1. 9. *The sum of any set of minimals is the negative of the sum of the remaining minimals.*

1. 10. *If m_1 and m_2 are two distinct minimals, then $m_1 m'_2 = m_1$.*

For, m'_2 is of the form $m_1 + a$, by 1. 9, and $m_1(m_1 + a) = m_1$.

More generally, we have:

1. 11. *If $m_1, m_2, \cdots, m_k$ are distinct minimals, then $m_k = m'_1 m'_2 \cdots m'_{k-1} m_k$.*

For, $m'_1, m'_2, \cdots, m'_{k-1}$ are each of the form $m_k + a$.

1. 12. *In the additive normal development of 1 with respect to the minimals, the non-0 constituents are precisely the minimals.*

By 1. 11 and 1. 8.

1. 13. *The minimals of $a + b$ consist of the minimals contained in a or b.*

1. 14. *The minimals of ab consist of the minimals common to a and b.*

1. 15. *A necessary and sufficient condition that a minimal m be a minimal of a is that $m < a$.*

For, $m < a$ is equivalent to the relation $a = a + m$. Hence the theorem, by 1. 1.

1.16. *If* $m < a + b$, *then* $m < a$ *or* $m < b$.

By 1.15 and 1.13.

1.17. $m \not< 0$.

For, if $m < 0$, then $0 = 0 + m = m$.

1.18. *Let* $ab = 0$; *if* $m < a$ *then* $m \not< b$.

For, if $m < a$ and $m < b$, then $m < ab$; hence $m < 0$, contrary to 1.17.

1.19. *If* $m \not< a$, *then* $ma = 0$.

For, if $m \not< a$, then m is not a minimal of a, by 1.15. Hence $ma = 0$, by 1.1, 1.4.

Further properties of the minimal elements will be stated with the help of the notion of "index" of an element and of other notions.

2. *Index of an element.* In a finite Boolean algebra B, by the *index* of an element a, denoted by $I(a)$, will be meant the number of minimals belonging to a. The index of the universe element 1, i.e., the number of minimals in B, will be called the index of B. A Boolean algebra of index k will be denoted by B_k.

We have, for indices, theorems 2.1-2.5 below. (In these theorems the operations between indices are, of course, arithmetic.)

2.1. *If* $ab = 0$, *then* $I(a + b) = I(a) + I(b)$.

For, a and b cannot have a common minimal, by 1.17. Hence the theorem, by 1.13.

The following two theorems are corollaries of 2.1.

2.2. *In an algebra* B_k, $I(a') = k - I(a)$.

2.3. *In an algebra* B_k, $I(a'b') = k - I(a + b)$.

Since if $b < a$ then $a = ab + ab' = b + ab'$, we have, by 2.1,

2.4. *If* $b < a$, *then* $I(ab') = I(a) - I(b)$.

Since $a + b = a + a'b = b + ab' = ab + ab' + a'b$, we have, by 2.1,

2.5. $I(a + b) = I(a) + I(a'b) = I(b) + I(ab') = I(a) + I(b) - I(ab)$.

3. *Addends.* The introduction of the notion of "addend" and of associated notions will enable us to view the minimal elements from another angle, and will also enable us to state conveniently some additional propositions for finite Boolean algebras.

In any Boolean algebra B, if there exists an x such that $b + x = a$, then b will be called an *addend* of a, and a a *complex* of b. Two elements a, b will be said to be *relative minimals*, if 0 is the only addend common to a and b. In a finite B, the minimals of a are, of course, addends of a, and a is a complex of its minimals. If a, $\neq 0$, has no addends other than 0 and a, then, by 1. 5, a is a minimal. In a finite B, an element c will be called the *least common complex* (*L. C. C.*) of a and b, if the minimals of c are the minimals belonging either to a or to b; an element c will be called the *highest common addend* (*H. C. A.*) of a and b, if the minimals of c are the minimals common to a and b.

3. 1. *In any Boolean algebra, a necessary and sufficient condition that b be an addend of a (or that a be a complex of b) is that $b < a$.*

For, $b < a$ is a necessary and sufficient condition that the equation $b + x = a$ have a solution.

3. 2. *In any Boolean algebra, a necessary and sufficient condition that a and b be relative minimals is that $ab = 0$.*

For, the pair of relations $x < a$, $x < b$ is equivalent to the single relation $x < ab$; and $ab = 0$ is a necessary and sufficient condition that $x < ab$ have the unique solution 0.

Theorems 3. 1 and 3. 2 enable us to restate the theorems of § 1 in terms of addends and related notions. Thus, 1. 16 and 1. 18 become respectively 3. 3 and 3. 4 following.

3. 3. *If a minimal m is an addend of $a + b$, then m is an addend of a or m is an addend of b.*

3. 4. *Let a and b be relative minimals; if m is an addend of a, then m is not an addend of b.*

In view of 1. 13 and 1. 14, we have:

3. 5. *$a + b = L. C. C.$ of a, b; $ab = H. C. A.$ of a, b.*

The following four theorems concern the addends of an element as constituting a Boolean algebra and an abelian group.

3. 6. *In any Boolean algebra B, the addends of an element u, $\neq 0$, form a Boolean algebra A with respect to the operations $+$ and $\times$, the elements 0, u, ux' of B serving in A as the "zero," the "universe," and the "negative of x" respectively.*

For, with respect to the operation $+$, the addends of u are related to u precisely as the elements of B are related to 1.

3. 7. *In any Boolean algebra B, the addends of an element u form an abelian group with respect to the operation o given by $x \circ y = xy' + x'y$.*

For, it can be verified, the operation o satisfies the following three conditions: (i) $a \circ b = b \circ a$; (ii) $(a \circ b) \circ c = a \circ (b \circ c)$; (iii) for any two addends a, b of u there is an addend x such that $a \circ x = b$.

If the algebra B in 3. 6 and 3. 7 be finite, these theorems become respectively Theorems 3. 8 and 3. 9 following.

3. 8. *In an algebra B_k, the addends of an element u, $\neq 0$, of index h, form a Boolean algebra B_h with respect to the operations $+$ and $\times$, the elements 0, u, ux' of B_k serving in B_h as the "zero," the "universe," and the "negative of x" respectively.*

3. 9. *In an algebra B_k, the addends of an element u, of index h, form an abelian group of order 2^h with respect to the operation o given by $x \circ y = xy' + x'y$.*

4. *Dual considerations. The maximals.* The foregoing considerations have, of course, their dual counterparts. Thus, corresponding to "minimal," "addend," "complex," "relative minimals," "least common complex," "highest common addend," we have, respectively, "maximal," "factor," "multiple," "relative maximals," "least common multiple," "highest common factor." The definitions of the notions in the latter set follow.

A *maximal* element μ of a finite Boolean algebra B is a non-1 constituent in the *complete* multiplicative normal development of 0. In any Boolean algebra, if there exists an x such that $bx = a$, *then* b is a *factor* of a, and a is a *multiple* of b; two elements a, b are *relative maximals* if 1 is the only factor common to a and b. In a finite Boolean algebra, an element c is the *least common multiple* (*L. C. M.*) of a and b, if the maximals of c are the maximals belonging either to a or to b; an element c is the *highest common factor* (*H. C. F.*) of a and b, if the maximals of c are the maximals common to a and b.

As samples of the duals of the foregoing propositions, I give propositions 4. 1-4. 5 below. These are the duals of 1. 1, 1. 4, 1. 6, 1. 10, 3. 1 respectively.

4. 1. *In a finite Boolean algebra, every element a, $\neq 1$, is expressible uniquely as a product of maximals.*

4. 2. *If $\mu_1, \mu_2, \cdots, \mu_k$ are the maximals of a finite Boolean algebra, then*

$$\mu_1\mu_2 \cdots \mu_k = 0, \qquad \mu_i + \mu_j = 1 \qquad [i \neq j].$$

4. 3. *In a finite Boolean algebra, a necessary and sufficient condition that a be a maximal is:*

(i) $a \neq 1$, (ii) $x \not> a$ $[x \neq 1,\ x \neq a]$.

4. 4. *If μ_1 and μ_2 are two distinct maximals, then $\mu_1 + \mu'_2 = \mu_1$.*

4. 5. *In any Boolean algebra, a necessary and sufficient condition that b be a factor of a (or that a be a multiple of b) is that $b > a$.*

In view of 3. 1, Theorem 4. 5 tells us:

4. 6. *In any Boolean algebra, a necessary and sufficient condition that a be a multiple of b is that a be an addend of b.*

5. *Relation between the minimals and the maximals.* As is to be expected, there is a very close relation between the minimal and the maximal elements of a finite Boolean algebra. This relation is given by the following theorem.

5. 1. *In a finite Boolean algebra, if m is a minimal then m' is a maximal.*

For, if m is a non-0 constituent in the complete additive normal development of 1, then m' is a non-1 constituent in the complete multiplicative normal development of 0.

As a consequence of 5. 1, in view of 1. 10, we have:

5. 2. *In a finite Boolean algebra, let m be a minimal and μ a maximal; then $m\mu = 0$, or else $m\mu = m$.*

Dually, we have:

5. 3. *In a finite Boolean algebra, let μ be a maximal and m a minimal; then $\mu + m = 1$, or else $\mu + m = \mu$.*

6. *Existence and construction of sub-algebras.* I shall now apply the properties of minimals to answer some questions concerning sub-algebras of a finite Boolean algebra.

We know that every Boolean algebra B has "embedded" in it as a sub-algebra the algebra B_1 of index 1, consisting of the elements 0, 1. Are there sub-algebras B_h $[h > 1]$ embedded in an algebra B_k $[k > h]$? If sub-algebras B_h exist, how construct these algebras? Again, given an algebra B_h embedded in a B_k $[k > h + 1]$, does there exist an algebra B_{h+1} embedded in B_k and embedding B_h? The answers to these questions will be found in the Theorems 6. 1-6. 3 following.

6.1. *Let $m_1, m_2, \cdots, m_k$ be the minimals of an algebra B_k $[k > 1]$; the $k-1$ elements*

$$m_1, m_2, \cdots, m_{k-2}, m_{k-1} + m_k$$

constitute a set of $k-1$ minimals for a sub-algebra B_{k-1} embedded in B_k.

The theorem is true by virtue of 1.7.

As a corollary of 6.1, we have:

6.2. *Every algebra B_k has embedded in it sub-algebras B_h of all indices h $[1 \leqq h \leqq k]$.*

Theorems 6.1 and 6.2 enable us to construct systematically all sub-algebras of a given finite Boolean algebra.

6.3. *Let $m_1, m_2, \cdots, m_h$ be the minimals of an algebra B_h embedded in an algebra B_k $[k > h]$; there is in B_k an element x such that for some minimal, say m_h, the $h+1$ elements*

$$m_1, m_2, \cdots, m_{h-1}, xm_h, x'm_h$$

constitute a set of minimals for an algebra B_{h+1} embedding B_h.

For, there exists in B_x some minimal of B_h, say m_h, for which there is an element x, $\neq 0$, $\neq m_h$, *such that* $x < m_h$; otherwise, $k = h$. Hence, there is in B_k an element x such that $xm_h = x$ $[x \neq 0, x \neq m_h]$. Hence, there is in B_k an element x such that $xm_h \neq 0$ $[x \neq 0, x \neq m_h]$. Also, for this x, we have $x'm_h \neq 0$, i. e. $m_h \not< x$; otherwise, $x = m_h$. Since $xm_h \neq 0$ and $x'm_h \neq 0$, the elements $m_1, m_2, \cdots, m_{h-1}, xm_h, x'm_h$ form, by 1.7, a set of minimals for an algebra B_{h+1}; and this algebra embeds B_h.

The reader can readily supply the duals of the above considerations.

7. *A problem in dichotomy.* In a Boolean algebra B, some of the constituents in the additive normal development of 1 with respect to h elements will, in general, vanish. The following problem then suggests itself.

PROBLEM. *In an algebra B_k, to find a set of h elements with respect to which none of the additive constituents of 1 vanish.*

The properties of the minimal elements established in the foregoing sections enable us to solve this problem readily. We observe, first, that in order that our problem have a solution, we must have, by 1.8, that $k \geqq 2^h$. Suppose $k \geqq 2^h$. Then, by 6.2, there exists in B_k a set of 2^h elements, $m_1, m_2, \cdots, m_{2^h}$,

which can serve as a set of minimals for a sub-algebra B_{2^h} embedded in B_k. The desired h elements are the elements a_i determined by the following $2^h - 1$ equations:

$$a_1 a_2 \cdot \cdot \cdot a_h = m_1,\ a_1 a_2 \cdot \cdot \cdot a'_h = m_2, \cdot \cdot \cdot,\ a'_1 a'_2 \cdot \cdot \cdot a_h = m_{2^h-1},$$

where the a-products are all different, and where $a'_1 a'_2 \cdot \cdot \cdot a'_h$ does not appear.

Thus, consider a Boolean algebra B_8. Let the minimals of B_8 be $m_1, m_2, \cdot \cdot \cdot, m_8$. A set of three elements with respect to which none of the additive constituents of 1 vanish are a_1, a_2, a_3 given by the following $2^3 - 1$ equations:

$$\begin{cases} a_1 a_2 a_3 = m_1, & a_1 a_2 a_3' = m_2, & a_1 a_2' a_3 = m_3, \\ a'_1 a_2 a_3 = m_4, & a_1 a'_2 a'_3 = m_5, & a'_1 a_2 a'_3 = m_6, \\ a'_1 a'_2 a_3 = m_7. \end{cases}$$

Indeed, noting that the sum of the products involving a_i affirmatively is a_i, we have:

$$\begin{cases} a_1 = m_1 + m_2 + m_3 + m_5, & a_2 = m_1 + m_2 + m_4 + m_6, \\ a_3 = m_1 + m_3 + m_4 + m_7. \end{cases}$$

The reader can supply the dual of the above problem and its solution.

8. *Arithmetic representations of finite Boolean algebras.* The properties of the minimal elements and related concepts deliver into our hands very simple arithmetic representations of finite Boolean algebras. These representations are given by the following considerations.

Let N_k be the class of 2^k numbers consisting of the number 1 and the products of h distinct primes taken from a set of k distinct primes $p_1, p_2, \cdot \cdot \cdot, p_k$ $[1 \leqq h \leqq k]$. Further, let $a \sqcap b$ be the least common multiple of a and b, and $a \Delta b$ the greatest common divisor of a and b. Then, no number of N_k contains a prime to a power higher than 1. Hence, the primes of N_k play with regard to $\sqcap$ and Δ the precise rôle that the minimals of B_k play with regard to Boolean addition, $\oplus$, and Boolean multiplication, $\odot$, respectively. Hence, *the arithmetic system* $(N_k, \sqcap, \Delta)$ *is simply isomorphic with the Boolean system* $(B_k, \oplus, \odot)$, $\sqcap$ *corresponding to* $\oplus$, *and* Δ *to* $\odot$.[5]

Dually, the primes of N_k play with regard to $\sqcap$ and Δ the precise rôle that the maximals of B_k play with regard to Boolean $\odot$ and $\oplus$ respectively.

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[5] The Boolean "zero," "universe," and "negative of a" correspond respectively to the N_k-elements 1, $v = p_1 \sqcap p_2 \sqcap \cdot \cdot \cdot \sqcap p_k$, $a_1 = v \div a$.

Hence, *the arithmetic system* $(N_k, \square, \Delta)$ *is simply isomorphic with the Boolean system* $(B_k, \odot, \oplus)$, $\square$ *corresponding to* $\odot$ *and* Δ *to* $\oplus$.[6]

It is to be observed that the representations of finite Boolean algebras given here enable us to "number" conveniently the elements of a finite Boolean algebra. Thus, to number the elements of an algebra B_3, we take for the three minimals in B_3 any three primes, for simplicity, the *first* three primes: 2, 3, 5. The numbered elements of B_3 are, then:

$$1, 2, 3, 5, 6\ (=2 \square 3),\ 10\ (=2 \square 5),\ 15\ (=3 \square 5),$$
$$30\ (=2 \square 3 \square 5 = v).^{7}$$

9. *New proof of the order theorem.* I shall close my discussion of finite Boolean algebras with a new proof of the theorem that *the order of a finite Boolean algebra is of the form* 2^k $[k > 0]$. This proof has interest in that no use is made in it of the minimal (or the maximal) elements. The proof follows.

Let A be a finite Boolean algebra of order n. Consider any group operation in A, say the operation o given by $a \circ b = ab' + a'b$. With respect to o the elements of A form an abelian group. Suppose, now, that A has a sub-group B of order 2^h $[h \geqq 1]$. Let $b_1, b_2, \cdots, b_{2^h}$ be the elements of B. Then, for any element c of A outside B, the 2^{h+1} elements

$$b_1, b_2, \cdots, b_{2^h}, b_1 \circ c, b_2 \circ c, \cdots, b_{2^h} \circ c$$

form in A an abelian group of order 2^{h+1}. Hence, since A is finite, the number n of its elements must be of the form 2^k $[k > 0]$. That is, we find that if A has any sub-group of order 2^h $[h \geqq 1]$, then the order of A is of the form 2^k $[k > 0]$. But A has the sub-group of order 2, consisting of the elements 0, 1. Hence the theorem.

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[6] The Boolean "universe," "zero," and "negative" correspond respectively to the N_k-elements 1, v, a_1.

[7] Compare Sheffer's representations of finite Boolean algebras, reproduced by Huntington, *Transactions of the American Mathematical Society*, vol. 35 (1933), p. 278.

LINEAR METRIC LEBESGUE SPACES.

By ANDREW C. BERRY.

The present paper is intended to serve as an introduction to a study of the elements of linear functional analysis in certain function-spaces whose metrics are given explicitly in terms of Lebesgue integrals. Spaces $L_{\phi,\psi}$, generalizations of spaces L_p of functions whose p-th powers are integrable, are introduced. Each such is here shown to be a complete linear metric space. Simple necessary and sufficient conditions for separability are found.

In the classical theory, spaces L_p, L_q, $L_{pq/(p+q)}$ are connected by an inequality due to Hölder. Using this fact, M. Riesz investigated convexity relations involving the individual members of families of such spaces. The indicated generalizations will be studied in a second paper.

Finally, by means of these new results, a third paper will extend the elementary theory of linear functionals from the classical symmetric case to the more nearly general case involving pairs of distinct spaces $L_{\phi,\psi}$.

I. THE GENERAL HYPOTHESES AND THE FUNDAMENTAL DEFINITION.

1.1. *Introduction.* It will be assumed that the reader is familiar with those laws of vector-algebra (dealing with addition of two elements and multiplication of an element by a complex number) which serve as postulates for a linear abstract space.[1] It will be assumed, further, that the reader is familiar with the details of the process of associating with each given Lebesgue-measurable function $f(x)$ every measurable function coinciding almost everywhere with $f(x)$, terming the totality thus formed a function-element, and extending algebraic operations and Lebesgue integration to these elements. Wherefore, the reader will allow us to avoid these precise constructions and to adopt in their stead the convenient, albeit paradoxical usage which considers each measurable function as an element and yet refuses to distinguish between elements $f(x)$ and $g(x)$ which, as functions, differ on a set of measure zero.

We shall investigate certain function-spaces $L_{\phi,\psi}$ each of which is determined by a metric-generating pair

[1] For complete details consult: N. Wiener, *Bulletin de la Société Mathématique de France*, t. 49 (1921), pp. 119-134; S. Banach, *Fundamenta Mathematicae*, t. 3 (1922), pp. 133-181; M. Fréchet, *Les espaces abstraits*, Paris, Gauthier-Villars (1928), particularly pp. 125-126; S. Banach, *Théorie des opérations linéaires*, Warsaw, Mathematical Seminar of the University of Warsaw (1932).

$$\phi(u), \qquad \psi(u)$$

of functions which, it will be understood throughout this paper, satisfy the following three requirements.

(1.11) $\phi(u)$ shall be defined for $0 \leqq u \leqq \infty$ as a non-negative, non-decreasing function of u, with $+\infty$ as a permissible value:

$$0 \leqq \phi(u) \leqq \phi(v) \leqq \infty, \qquad (0 \leqq u < v \leqq \infty).$$

Furthermore, it shall be required that

$$\phi(0) = 0, \quad \phi(1) = 1, \quad \phi(\infty) = \lim_{u \to \infty} \phi(u).$$

(1.12) $\psi(u)$ shall be defined for $0 < u < \infty$ as a non-negative, non-decreasing function of u, with $+\infty$ as a permissible value:

$$0 \leqq \psi(u) \leqq \psi(v) \leqq \infty, \qquad (0 < u < v < \infty).$$

Furthermore, it shall be required that

$$0 < \psi(1) < \infty.$$

(1.13) The inequality

$$\psi(sw)\psi(tw)\phi(su + tv) \leqq \psi(w)\{s\psi(tw)\phi(u) + t\psi(sw)\phi(v)\}$$

shall be satisfied whenever all the functional values involved are finite and

$$0 < s, \quad 0 < t, \quad s + t = 1, \quad 0 \leqq u < \infty, \quad 0 \leqq v < \infty, \quad 0 < w < \infty.$$

Definition 1.14. $L_{\phi,\psi}$ will denote the totality of complex-valued Lebesgue-measurable functions, $f(x)$, of the single real variable x, $-\infty < x < \infty$, for each of which there exists, as a finite number,

$$l_{\phi,\psi}(f) = \inf \epsilon > 0 \text{ such that } \int \phi\left(\frac{|f(x)|}{\epsilon}\right) dx \leqq \psi(\epsilon),$$

the integral being a Lebesgue integral [2] extended over $(-\infty, \infty)$. If, for a given ϵ, it is true that $\psi(\epsilon) = \infty$, then the inequality of this definition is to be considered satisfied for this ϵ.

1.2. Discussion of the general hypotheses.

For convenience, we introduce the following standard notation.

[2] It is known that a monotone function of a measurable function of a real variable is itself a measurable function of the variable. See, for example, Carathéodory, *Vorlesungen über reelle Funktionen*, 2nd ed. (1927), B. G. Teubner, Berlin, §§ 348, 351.

$$\begin{cases} \phi(u+) = \lim\limits_{\substack{\epsilon\to 0 \\ (\epsilon>0)}} \phi(u+\epsilon),\ \psi(u+) = \lim\limits_{\substack{\epsilon\to 0 \\ (\epsilon>0)}} \psi(u+\epsilon), & (0 \leqq u < \infty). \\ \phi(u-) = \lim\limits_{\substack{\epsilon\to 0 \\ (0<\epsilon<u)}} \phi(u-\epsilon),\ \psi(u-) = \lim\limits_{\substack{\epsilon\to 0 \\ (0<\epsilon<u)}} \psi(u-\epsilon), & (0 < u < \infty). \end{cases}$$

THEOREM 1.21. *If* $\phi(0+) > 0$, *then* $\psi(0+) = 0$.

Proof. In (1.13) we set $u = 0$, $v = 1$, $w = 1$, and so find

$$\psi(s)\psi(t)\phi(t) \leqq \psi(1)[t\psi(s)] \leqq [\psi(1)]^2 t.$$

Noting that $\psi(t) \leqq \psi(s)$ when $0 < t \leqq \frac{1}{2}$, we conclude that

$$[\psi(t)]^2 \leqq \frac{[\psi(1)]^2}{\phi(0+)}\, t, \qquad (0 < t \leqq \tfrac{1}{2}).$$

This result establishes our theorem.

We shall now show that the two requirements: $\phi(1) = 1$, $0 < \psi(1) < \infty$, introduce no essential loss of generality. We note, in passing, that the condition $\phi(0) = 0$ makes certain that the null element (i.e., "the" function which is zero almost everywhere) will have the length zero. That there may be no second element of length zero it is necessary to require both that $\phi(u)$ be not identically zero in its domain of definition and that $\psi(u)$ possess at least one finite value. Thus, there must exist u_1 and v_1 such that

$$0 < u_1 < \infty, \qquad 0 < \phi(u_1);$$
$$0 < v_1 < \infty, \qquad \psi(v_1) < \infty.$$

Next, it is desirable that $L_{\phi,\psi}$ fail to be a null space (i.e., a space consisting solely of the null element). One necessary condition for such failure (since it has just been shown that $\phi(u_1) > 0$) is that $\psi(u)$ have at least one positive value. Thus, there must exist v_2 such that

$$0 < v_2 < \infty, \qquad 0 < \psi(v_2).$$

A second condition, necessary in the case in which $\psi(u)$ is everywhere finite, is: $\phi(0+) < \infty$.

Another degenerate type of space $L_{\phi,\psi}$, also sufficiently trivial to warrant rejection, is that which contains as elements all measurable functions $f(x)$ and assigns to all its elements, other than the null element, one constant length l, $0 < l < \infty$. The conditions: $\phi(0+) = \infty$, $\psi(u) = \infty$ for some u, are separately necessary and collectively sufficient that $L_{\phi,\psi}$ be such a space (with l defined by the relations $\psi(l-) < \infty$, $\psi(l+) = \infty$).

In view of the foregoing remarks we finally must require $\phi(0+)$ to be finite. Thus, there must exist u_2 such that

$$0 < u_2 < \infty, \qquad \phi(u_2) < \infty.$$

On combining our several results we readily demonstrate that there must exist u_0 and v_0 such that

$$\begin{aligned} &0 < u_0 < \infty, \quad 0 < \phi(u_0 +), \quad \phi(u_0 -) < \infty; \\ &0 < v_0 < \infty, \quad 0 < \psi(v_0 +), \quad \psi(v_0 -) < \infty. \end{aligned}$$

A brief consideration of the fundamental definition shows that we may alter, if necessary, the values $\phi(u_0)$ and $\psi(v_0)$ so that

$$0 < \phi(u_0) < \infty, \qquad 0 < \psi(v_0) < \infty,$$

without altering the remaining values of these functions, and so without altering the space $L_{\phi,\psi}$.

Now, introduce the functions

$$\phi_1(u) = \frac{\phi(u_0 u)}{\phi(u_0)}, \qquad \psi_1(u) = \frac{\psi(v_0 u)}{\phi(u_0)}.$$

We see, firstly, that

$$\phi_1(1) = 1, \qquad 0 < \psi_1(1) < \infty,$$

and, secondly, that

$$l_{\phi,\psi}(u_0 v_0 f) = v_0 l_{\phi_1,\psi_1}(f).$$

The two spaces $L_{\phi,\psi}$ and L_{ϕ_1,ψ_1} are essentially equivalent, the one being merely a homogeneous enlargement of the other (partially from the point of view of the metric assigned to a given element and partially from the point of view of the elements themselves). Thus, the aim of the present discussion has been attained.

1. 3. Examples of spaces $L_{\phi,\psi}$.

THEOREM 1. 31. *If $\phi(u)$ satisfies* (1. 11) *and if $\psi(u) = u$, $(0 < u < \infty)$, then the general hypotheses are fulfilled for the pair $\phi(u)$, $\psi(u)$.*

Proof. The function $\psi(u) = u$ satisfies (1. 12) and reduces (1. 13) to

$$\phi(su + tv) \leqq \phi(u) + \phi(v),$$

which automatically is satisfied since $(su + tv)$ is intermediate in value between u and v and since $\phi(u)$ is non-negative and monotone.

THEOREM 1. 32. *If $\phi(u)$ satisfies* (1. 11) *and is a "convex" function of u (i. e., satisfies the inequality: $\phi(su + tv) \leqq s\phi(u) + t\phi(v)$, whenever $0 < s$, $0 < t$, $s + t = 1$), and if $\psi(u) = constant$ $(0 < const < \infty)$ for $0 < u < \infty$, then the pair $\phi(u), \psi(u)$ is satisfactory.*

Proof. Immediate.

Example 1.33. By 1.32, the pair

$$\begin{cases} \phi(u) = u^p \text{ (for some } p \text{ such that } 1 \leqq p < \infty), & (0 \leqq u \leqq \infty), \\ \psi(u) = 1, \quad (0 < u < \infty), \end{cases}$$

is known to satisfy the general hypotheses. A simple calculation yields:

$$l_{\phi,\psi}(f) = \left\{ \int | f(x) |^p dx \right\}^{1/p}.$$

Thus, the corresponding $L_{\phi,\psi}$ is the familiar space of measurable functions whose p-th powers are integrable.[3] This is known, in particular, to be a separable, complete, linear metric space with homogeneous metric.

Example 1.34. Consider the pair

$$\begin{cases} \phi(u) = u^q \text{ (for some } q \text{ such that } 0 < q < 1), & (0 \leqq u \leqq \infty), \\ \psi(u) = u^{1-q}, \quad (0 < u < \infty). \end{cases}$$

Here, (1.13) becomes:

$$(su + tv)^q \leqq (su)^q + (tv)^q.$$

The reader will establish the validity of this inequality. The corresponding space later will be seen to be separable, complete, and linear metric. However, the metric

$$l_{\phi,\psi}(f) = \int | f(x) |^q dx$$

is not homogeneous.

Example 1.35. For the, by 1.32, satisfactory pair

$$\begin{cases} \phi(u) = \lim\limits_{p \to \infty} u^p = \begin{cases} 0, & (0 \leqq u < 1), \\ 1, & (u = 1), \\ \infty, & (1 < u \leqq \infty), \end{cases} \\ \psi(u) = 1, \quad (0 < u < \infty), \end{cases}$$

we find

$$\begin{aligned} l_{\phi,\psi}(f) &= \text{upper measurable bound of } | f(x) | \\ &= \inf \epsilon > 0 \text{ such that meas } \{ | f(x) | > \epsilon \} = 0. \end{aligned}$$

This metric is homogeneous and the corresponding $L_{\phi,\psi}$ is the familiar non-separable, complete, linear metric space of measurably bounded functions.

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[3] F. Riesz, "Über Systeme integrierbarer Funktionen," *Mathematische Annalen*, Bd. 69 (1910), pp. 449-497; Banach, *Théorie des opérations linéaires*, cited above.

Example 1.36. The, by 1.31, satisfactory pair

$$\begin{cases} \phi(u) = \lim\limits_{\substack{q\to 0 \\ (q>0)}} u^q = \begin{cases} 0, & (u=0), \\ 1, & (0<u<\infty), \end{cases} & \phi(\infty)=1, \\ \psi(u) = u, \quad (0<u<\infty), \end{cases}$$

yields the non-homogeneous metric

$$l_{\phi,\psi}(f) = \text{meas}\,\{\,|f(x)|>0\}.$$

Example 1.37. The pair

$$\begin{cases} \phi(u) = \begin{cases} 0, & (0 \leqq u < 1), \\ 1, & (1 \leqq u \leqq \infty), \end{cases} \\ \psi(u) = \quad u, \quad (0<u<\infty), \end{cases}$$

determines the metric

$$l_{\phi,\psi}(f) = \inf \epsilon > 0 \text{ such that } \text{meas}\,\{\,|f(x)|>\epsilon\} \leqq \epsilon.$$

The corresponding space $L_{\phi,\psi}$ has been discussed by us elsewhere.[4]

II. Each $L_{\phi,\psi}$ is a Complete Linear Metric Space.

If $f(x)$ is an element of a space $L_{\phi,\psi}$ having the property S (which property will be defined in 3.11), then $f(x)$ must be finite almost everywhere. For a general $L_{\phi,\psi}$, however, a linear combination of elements, $af(x)+bg(x)$, may involve the indeterminate forms: $\infty-\infty$, $0\cdot\infty$. On the set of indeterminate values the linear combination may be defined according to our pleasure by equating it to any measurable function. The modifications thus necessitated in the linear algebra are well known.

2.1. Preliminary estimates.

Theorem 2.11. *If $f(x)$ is an element of a given $L_{\phi,\psi}$ and if $\epsilon > l_{\phi,\psi}(f)$, then the inequality of* 1.14 *will be satisfied for this $f(x)$ and this ϵ.*

Proof. By 1.14, there exists ϵ_1, $l_{\phi,\psi}(f) \leqq \epsilon_1 < \epsilon$, such that

$$\int \phi\left(\frac{|f(x)|}{\epsilon_1}\right) dx \leqq \psi(\epsilon_1).$$

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[4] Berry, *Proceedings of the National Acad. of Sciences*, vol. 17 (1931), pp. 456-459. I take this opportunity to make a correction. In this note I stated that the metric

$$l(f) = \inf_{(\epsilon>0)} (\epsilon + \text{meas}\,\{|f(x)|>\epsilon\}),$$

introduced by M. Fréchet ("Sur divers modes de convergence d'une suite de fonctions d'une variable," *Bulletin of the Calcutta Mathematical Society*, vol. 11 (1921), pp. 187-206; *Les espaces abstraits*, cited above, p. 91) was said by him not to satisfy the triangle inequality 2.22. Secondly, I impliedly agreed with this supposedly quoted opinion.

By (1. 11) and (1. 12) the desired inequality,

$$\int \phi\left(\frac{|f(x)|}{\epsilon}\right) dx \leqq \psi(\epsilon),$$

follows.

THEOREM 2. 12. *If $f(x)$ is an element of a given space $L_{\phi,\psi}$, if $g(x)$ is measurable, and if $|g(x)| \leqq |f(x)|$, almost everywhere, then $g(x)$ is an element of $L_{\phi,\psi}$, and $l_{\phi,\psi}(g) \leqq l_{\phi,\psi}(f)$.*

Proof. By (1. 11), it is seen that the inequality of 1. 14, if satisfied for the given $f(x)$ and for a given $\epsilon > 0$, also will be satisfied for $g(x)$ and the same ϵ. This fact establishes the theorem.

THEOREM 2. 13. *If $f(x)$ is an element of a given $L_{\phi,\psi}$, if $0 \leqq l_{\phi,\psi}(f) < \frac{1}{4}$, and if u_0, $0 \leqq u_0 < \infty$ is such that $0 < \phi(u_0+)$, then*

$$\text{meas}\{|f(x)| > u_0\sqrt{l_{\phi,\psi}(f)}\} \leqq \frac{\psi(1)}{\phi(u_0+)}\sqrt{l_{\phi,\psi}(f)}.$$

Proof. Let ϵ and η be chosen so that

$$0 \leqq l_{\phi,\psi}(f) < \epsilon < \tfrac{1}{4}, \qquad 0 < \eta < \infty,$$

and let these values be kept fixed throughout the discussion of the following three cases.

Case A. $\psi(\epsilon) = 0$.

Here, by 2. 11,

$$\phi\left(\frac{|f(x)|}{\epsilon}\right) = 0, \text{ almost everywhere, } (-\infty < x < \infty).$$

Since

$$\phi\left(\frac{u_0+\eta}{\epsilon^{\frac{1}{2}}}\right) \geqq \phi(u_0+) > 0,$$

it follows that

$$\text{meas}\{|f(x)| > (u_0+\eta)\epsilon^{\frac{1}{2}}\} = 0.$$

Case B. $\phi\left(\frac{u_0+\eta}{\epsilon^{\frac{1}{2}}}\right) = \infty$.

Here, since 2. 11 yields:

$$\int \phi\left(\frac{|f(x)|}{\epsilon}\right) dx \leqq \psi(\epsilon) \leqq \psi(1) < \infty,$$

we see that

$$\text{meas}\{|f(x)| > (u_0+\eta)\epsilon^{\frac{1}{2}}\} = 0.$$

Case C. $\psi(\epsilon) > 0$ and $\phi\left(\frac{u_0+\eta}{\epsilon^{1/2}}\right) < \infty$.

Here we note that

$$0 < \phi(u_0+) \leqq \phi(u_0+\eta) \leqq \phi\left(\frac{u_0+\eta}{\epsilon^{1/2}}\right) < \infty,$$

$$0 < \psi(\epsilon) \leqq \psi(\epsilon^{1/2}) \leqq \psi(1-\epsilon^{1/2}) \leqq \psi(1) < \infty.$$

Thus, if in (1.13) we set $u=0$, $v=\left(\frac{u_0+\eta}{\epsilon^{1/2}}\right)$, $w=1$, $s=1-\epsilon^{1/2}$, $t=\epsilon^{1/2}$, we find that

$$1 \leqq \frac{\psi(1)}{\phi(u_0+\eta)}\,\epsilon^{1/2}\,\frac{\phi\left(\frac{u_0+\eta}{\epsilon^{1/2}}\right)}{\psi(\epsilon)}.$$

Whence,

$$\begin{aligned}
&\text{meas}\,\{|f(x)| > (u_0+\eta)\epsilon^{1/2}\} \\
&\leqq \frac{\psi(1)}{\phi(u_0+\eta)}\,\epsilon^{1/2}\,\frac{\phi\left(\frac{u_0+\eta}{\epsilon^{1/2}}\right)\text{meas}\,\{|f(x)| > (u_0+\eta)\epsilon^{1/2}\}}{\psi(\epsilon)} \\
&\leqq \frac{\psi(1)}{\phi(u_0+\eta)}\,\epsilon^{1/2}\,\frac{\int \phi\left(\frac{|f(x)|}{\epsilon}\right)dx}{\psi(\epsilon)} \leqq \frac{\psi(1)}{\phi(u_0+\eta)}\,\epsilon^{1/2}.
\end{aligned}$$

In all cases we have found that the inequality

$$\text{meas}\,\{|f(x)| > (u_0+\eta)\epsilon^{1/2}\} \leqq \frac{\psi(1)}{\phi(u_0+\eta)}\,\epsilon^{1/2}$$

holds for each sufficiently small $\epsilon > l_{\phi,\psi}(f)$ and for each $\eta > 0$. Considering only such values of ϵ and η, but requiring

$$\epsilon \to l_{\phi,\psi}(f), \qquad \eta \to 0,$$

we see that the set $\{|f(x)| > (u_0+\eta)\epsilon^{1/2}\}$ *expands* to the limiting set $\{|f(x)| > u_0\sqrt{l_{\phi,\psi}(f)}\}$. Under such circumstances we know that meas lim = lim meas. Therefore,

$$\text{meas}\,\{|f(x)| > u_0\sqrt{l_{\phi,\psi}(f)}\} \leqq \frac{\psi(1)}{\phi(u_0+)}\sqrt{l_{\phi,\psi}(f)}.$$

2.2. Proof that each $L_{\phi,\psi}$ is a linear metric space.

THEOREM 2.21. *A necessary and sufficient condition that* $l_{\phi,\psi}(f) = 0$ *is that* $f(x) = 0$ *almost everywhere, i.e., that* f *be the null element.*

Proof of sufficiency. Let $f(x) = 0$ almost everywhere. Then for each $\epsilon > 0$,

$$\int \phi\left(\frac{|f(x)|}{\epsilon}\right) dx = 0 \leqq \psi(\epsilon).$$

By definition 1. 14, $l_{\phi,\psi}(f) = 0$.

Proof of necessity. Let $l_{\phi,\psi}(f) = 0$. Since $\phi(1+) > 0$, it follows from 2. 13 that

$$\text{meas}\,\{|f(x)| > 0\} = 0,$$

which means precisely that $f(x) = 0$, almost everywhere.

THEOREM 2. 22. *If $f(x)$ and $g(x)$ are both elements of a given $L_{\phi,\psi}$, then the "triangle inequality,"*

$$l_{\phi,\psi}(f+g) \leqq l_{\phi,\psi}(f) + l_{\phi,\psi}(g),$$

is valid and implies that $f(x) + g(x)$ also is an element of $L_{\phi,\psi}$.

Proof. Let ϵ and η be chosen so that

$$l_{\phi,\psi}(f) < \epsilon < \infty, \quad l_{\phi,\psi}(g) < \eta < \infty,$$

and let these values be kept fixed during the discussion of the three following cases.

Case A. $\psi(\epsilon+\eta) = \infty$.

Here we are to regard the inequality

$$\int \phi\left(\frac{|f(x)+g(x)|}{\epsilon+\eta}\right) dx \leqq \psi(\epsilon+\eta)$$

as satisfied.

Case B. At least one of the values $\psi(\epsilon)$, $\psi(\eta)$ is zero.

Without loss of generality we may assume that $\psi(\epsilon) = 0$. This, in virtue of 2. 11, implies that

$$\phi\left(\frac{|f(x)|}{\epsilon}\right) = 0, \text{ almost everywhere, } (-\infty < x < \infty).$$

Since $\phi(u)$ is non-decreasing, we conclude that

$$\phi\left(\frac{|f(x)+g(x)|}{\epsilon+\eta}\right) \leqq \phi\left(\frac{|f(x)|+|g(x)|}{\epsilon+\eta}\right)$$

$$\leqq \max\left[\phi\left(\frac{|f(x)|}{\epsilon}\right), \phi\left(\frac{|g(x)|}{\eta}\right)\right] = \phi\left(\frac{|g(x)|}{\eta}\right), \text{ almost everywhere.}$$

Thus,

$$\int \phi\left(\frac{|f(x)+g(x)|}{\epsilon+\eta}\right)dx \leqq \int \phi\left(\frac{|g(x)|}{\eta}\right)dx \leqq \psi(\eta) \leqq \psi(\epsilon+\eta).$$

Case C. $0<\psi(\epsilon)$, $0<\psi(\eta)$ and $\psi(\epsilon+\eta)<\infty$.

If, in (1.13), we set

$$u=\frac{|f(x)|}{\epsilon},\quad v=\frac{|g(x)|}{\eta},\quad w=\epsilon+\eta,\quad s=\frac{\epsilon}{\epsilon+\eta},\quad t=\frac{\eta}{\epsilon+\eta},$$

we find that

$$\phi\left(\frac{|f(x)+g(x)|}{\epsilon+\eta}\right) \leqq \phi\left(\frac{|f(x)|+|g(x)|}{\epsilon+\eta}\right)$$

$$\leqq \psi(\epsilon+\eta)\left\{\frac{\epsilon}{\epsilon+\eta}\,\frac{\phi\left(\frac{|f(x)|}{\epsilon}\right)}{\psi(\epsilon)}+\frac{\eta}{\epsilon+\eta}\,\frac{\phi\left(\frac{|g(x)|}{\eta}\right)}{\psi(\eta)}\right\},\ \text{almost everywhere.}$$

Since 2.11 yields:

$$\int \phi\left(\frac{|f(x)|}{\epsilon}\right)dx \leqq \psi(\epsilon),\ \text{and}\ \int \phi\left(\frac{|g(x)|}{\eta}\right)dx \leqq \psi(\eta),$$

it follows that

$$\int \phi\left(\frac{|f(x)+g(x)|}{\epsilon+\eta}\right)dx \leqq \psi(\epsilon+\eta).$$

Collecting results, we find that this last inequality is valid in all cases. We conclude, therefore, that

$$l_{\phi,\psi}(f+g) \leqq \epsilon+\eta$$

for all $\epsilon > l_{\phi,\psi}(f)$ and all $\eta > l_{\phi,\psi}(g)$. This is possible if and only if

$$l_{\phi,\psi}(f+g) \leqq l_{\phi,\psi}(f)+l_{\phi,\psi}(g).$$

THEOREM 2.23. *If $f(x)$ is in $L_{\phi,\psi}$ and if a is a finite complex number, then $af(x)$ is in $L_{\phi,\psi}$, and*

$$\begin{cases} l_{\phi,\psi}(af) \leqq |a|\, l_{\phi,\psi}(f), & \text{when } |a|>1,\\ l_{\phi,\psi}(af) = \quad l_{\phi,\psi}(f), & \text{when } |a|=1,\\ l_{\phi,\psi}(af) \leqq \quad l_{\phi,\psi}(f), & \text{when } |a|<1. \end{cases}$$

In the special case in which $\psi(u)=$ constant, $(0<u<\infty)$, these relations can be strengthened to the equality $l_{\phi,\psi}(af) = |a|\, l_{\phi,\psi}(f)$ valid for all a; i. e., the metric is homogeneous in this case.

Proof. We need only consider the case: $|a| > 0$.

When $\psi(u) =$ constant, the inequalities

$$\int \phi \frac{|f(x)|}{\epsilon}\, dx \leqq \psi(\epsilon), \quad \int \phi \frac{|af(x)|}{|a|\,\epsilon}\, dx \leqq \psi(|a|\,\epsilon),$$

have their left members equal and their right members equal, and so are equivalent in the sense that whenever one is valid so also is the other. By 1.14, then, $l_{\phi,\psi}(af) = |a|\, l_{\phi,\psi}(f)$, for all a.

In general we see that these same two inequalities are equivalent only when $|a| = 1$, but that the validity of the first implies that of the second whenever $|a| > 1$. These facts establish, respectively, the second and first relations of the theorem. The third relation is an immediate consequence of 2.12.

2.3. An existence theorem.

THEOREM 2.31. *If $g_1(x), g_2(x), \cdots$, are elements of a given $L_{\phi,\psi}$; if (for a given constant, T, independent of n) it is true that $g_n(x) = 0$ whenever $|x| > T$, $(n = 1, 2, \cdots)$; if*

$$|g_{n+1}(x) - g_n(x)| \leqq 2^{-n-1}, \text{ for all } x \text{ and all } n;$$

and if for each p, $(p = 1, 2, \cdots)$,

$$l_{\phi,\psi}(g_m - g_n) \leqq 2^{-2p-2}, \text{ when } m \geqq p \text{ and } n \geqq p;$$

then there exists an element $g(x)$ in $L_{\phi,\psi}$ such that

$$\lim_{n\to\infty} l_{\phi,\psi}(g - g_n) = 0.$$

Proof. We note first that, since

$$g_n(x) = g_1(x) + \{g_2(x) - g_1(x)\} + \cdots + \{g_n(x) - g_{n-1}(x)\},$$

there exists

$$g(x) = \lim_{n\to\infty} g_n(x) = g_1(x) + \{g_2(x) - g_1(x)\} + \cdots,$$

defined for all x as the sum of a uniformly and absolutely convergent series. In particular, we see that

$$|g(x) - g_n(x)| \leqq |g_{n+1}(x) - g_n(x)| + |g_{n+2}(x) - g_{n+1}(x)| + \cdots \leqq 2^{-n},$$

for all x and all n.

Now let $\epsilon > 0$ be given.

Case A. $\psi(\epsilon) = 0$.

Here, determine n_ϵ so that

$$2^{-n_\epsilon} < \epsilon.$$

Since $2^{-n-1} > 2^{-2n-2}$ when $n = 1, 2, \cdots$, theorem 2.11 yields:

$$\phi\left(\frac{|\, g_{n+1}(x) - g_n(x)\,|}{2^{-n-1}}\right) = 0, \text{ almost everywhere (for each } n \geqq n_\epsilon).$$

Next we observe that, for all x,

$$\frac{|\, g(x) - g_n(x)\,|}{\epsilon} \leqq \frac{|\, g_{n+1}(x) - g_n(x)\,|}{2^{-n-1}} + \frac{|\, g_{n+2}(x) - g_{n+1}(x)\,|}{2^{-n-2}} + \cdots$$

$$\leqq \sup_{p \geqq n} \frac{|\, g_{p+1}(x) - g_p(x)\,|}{2^{-p-1}}, \text{ whenever } n \geqq n_\epsilon.$$

Since the union of a denumerable infinitude of sets of measure zero is likewise a set of measure zero, and since $\phi(u)$ is non-decreasing, we conclude that

$$\phi\left(\frac{|\, g(x) - g_n(x)\,|}{\epsilon}\right) = 0, \text{ almost everywhere (for each } n \geqq n_\epsilon).$$

This implies that

$$l_{\phi,\psi}(g - g_n) \leqq \epsilon, \text{ for all } n \geqq n_\epsilon.$$

Case B. $\psi(\epsilon) > 0$, $\phi(0+) = 0$.

Here, determine n_ϵ so that

$$2T\,\phi\left(\frac{2^{-n_\epsilon}}{\epsilon}\right) \leqq \psi(\epsilon).$$

Then, for all $n \geqq n_\epsilon$,

$$\int \phi\left(\frac{|\, g(x) - g_n(x)\,|}{\epsilon}\right) dx \leqq 2T\,\phi\left(\frac{2^{-n}}{\epsilon}\right) \leqq \psi(\epsilon).$$

That is,

$$l_{\phi,\psi}(g - g_n) \leqq \epsilon, \text{ for all } n \geqq n_\epsilon.$$

Case C. $\psi(\epsilon) > 0$, $\phi(0+) > 0$.

Here, determine n_ϵ so that

$$2^{-n_\epsilon} < \min\left[\epsilon, \frac{\phi(0+)\psi(\epsilon)}{\psi(1)}\right].$$

By 2.13, with $u_0 = 0$,

$$\text{meas}\,\{|\, g_{n+1}(x) - g_n(x)\,| > 0\} \leqq \frac{\psi(1)}{\phi(0+)}\, 2^{-n-1}, \quad (n = 1, 2, \cdots).$$

Whence

$$\int \phi\left(\frac{|g(x)-g_n(x)|}{\epsilon}\right)dx \leqq \phi\left(\frac{2^{-n}}{\epsilon}\right)\text{meas}\{|g(x)-g_n(x)|>0\}$$
$$\leqq \phi(1)\sum_{p=n}^{\infty}\text{meas}\{|g_{p+1}(x)-g_p(x)|>0\}$$
$$\leqq \frac{\psi(1)}{\phi(0+)}2^{-n} \leqq \psi(\epsilon), \text{ for all } n \geqq n_\epsilon.$$

That is,

$$l_{\phi,\psi}(g-g_n) \leqq \epsilon, \text{ for all } n \geqq n_\epsilon.$$

Thus we have shown that the function $g(x)$ constructed at the outset of our proof meets the requirement:

$$\lim_{n\to\infty} l_{\phi,\psi}(g-g_n) = 0.$$

Since $g = (g-g_n)+g_n$ and since $L_{\phi,\psi}$ is linear we see that this same $g(x)$ is also an element of $L_{\phi,\psi}$.

2.4. On non-decreasing sequences of non-negative functions.

THEOREM 2.41. *If $h_1(x), h_2(x), \cdots$ are elements of a given $L_{\phi,\psi}$, if for each j, $(j=1,2,\cdots)$.*

$$0 \leqq h_j(x) \leqq h_{j+1}(x), \quad \textit{almost everywhere}, \quad (-\infty < x < \infty),$$

and if there is a constant, M, independent of j, such that

$$l_{\phi,\psi}(h_j) \leqq M, \qquad (j=1,2,\cdots),$$

then

$$h(x) = \lim_{j\to\infty} h_j(x) \leqq \infty$$

exists almost everywhere, is an element of $L_{\phi,\psi}$, and is such that

$$l_{\phi,\psi}(h) \leqq M.$$

Proof. Since the denumerably many inequalities, $h_j(x) \leqq h_{j+1}(x)$, are valid simultaneously except at most on a set of measure zero, it follows that $h(x)$ exists almost everywhere.

Let $\epsilon > M$ be given. Determine some ϵ_0 such that $\epsilon > \epsilon_0 > M$. By 2.11,

$$\int \phi\left(\frac{h_j(x)}{\epsilon_0}\right)dx \leqq \psi(\epsilon_0) \leqq \psi(\epsilon), \qquad (j=1,2,\cdots).$$

As $j\to\infty$, the non-negative integrand increases monotonely. Hence,

$$\int \left\{\lim_{j\to\infty}\phi\left(\frac{h_j(x)}{\epsilon_0}\right)\right\}dx = \lim_{j\to\infty}\int \phi\left(\frac{h_j(x)}{\epsilon_0}\right)dx.$$

But, by (1.11),

$$\lim_{j\to\infty}\phi\left(\frac{h_j(x)}{\epsilon_0}\right) \geqq \phi\left(\frac{h(x)}{\epsilon}\right), \text{ almost everywhere.}$$

Whence,

$$\int \phi\left(\frac{h(x)}{\epsilon}\right) dx \leqq \psi(\epsilon)$$

for each $\epsilon > M$. This implies that $l_{\phi,\psi}(h) \leqq M$, and so establishes the theorem.

Note 2.42. Under the hypotheses of theorem 2.41 we might expect to conclude further that

$$\lim_{j\to\infty} l_{\phi,\psi}(h - h_j) = 0.$$

We shall see indeed, in theorem 3.22, that this conclusion can be guaranteed when $L_{\phi,\psi}$ has the property S defined in 3.11. However, if $L_{\phi,\psi}$ fails to have the property S, then the conclusion under consideration can be shown to be false by means of the functions constructed in theorem 3.41. The details will be discussed in note 3.23.

2.5. Proof that each $L_{\phi,\psi}$ is complete.

THEOREM 2.51. *If $f_1(x), f_2(x), \cdots$ are elements of a given $L_{\phi,\psi}$ and if*

$$\lim_{m,n\to\infty} l_{\phi,\psi}(f_m - f_n) = 0,$$

then there exists an element $f(x)$ of $L_{\phi,\psi}$ such that

$$\lim_{n\to\infty} l_{\phi,\psi}(f - f_n) = 0.$$

Proof. If the theorem can be established for a sub-sequence of the given sequence of elements, then a simple application of 2.22 will establish the theorem for the given sequence. Therefore, we need prove our theorem in no more general a case than the following in which we assume that

$$l_{\phi,\psi}(f_m - f_n) \leqq 2^{-2p-2}, \text{ whenever } m \geqq p \text{ and } n \geqq p, \quad (p = 1, 2, \cdots).$$

This implies, by 2.13 with $u_0 = 1$, that for each p, $(p = 1, 2, \cdots)$,

$$\text{meas } \{|f_m(x) - f_n(x)| > 2^{-p-1}\} \leqq \psi(1)\, 2^{-p-1}, \quad (m \geqq p, n \geqq p).$$

For each j, $(j = 1, 2, \cdots)$, let the corresponding set

$$\sum_{p=j}^{\infty} \{|f_{p+1}(x) - f_p(x)| > 2^{-p-1}\}$$

be denoted by E_j and let the complement with respect to the entire interval, $-\infty < x < \infty$, be denoted by $\bar{E}_j$. We observe that

$$\text{meas } E_j \leqq \sum_{p=j}^{\infty} \text{meas } \{|f_{p+1}(x) - f_p(x)| > 2^{-p-1}\} \leqq \psi(1)\, 2^{-j}, \quad (j = 1, 2, \cdots),$$

and that each set E_j contains the corresponding set E_{j+1}. Finally, we see that x is in $\bar{E}_j$ if and only if *all* of the inequalities

$$|f_{p+1}(x) - f_p(x)| \leqq 2^{-p-1}, \qquad (p = j, j+1, \cdots),$$

are valid.

Let us temporarily fix j. Construct the functions:

$$g_n(x) = \begin{cases} f_n(x), & \text{if, simultaneously, } x \text{ is in } \bar{E}_j \text{ and } |x| \leqq j, \\ 0, & \text{otherwise,} \end{cases} \qquad (n = 1, 2, \cdots).$$

By 2.12, the sequence $g_1(x), g_2(x), \cdots$ is seen to satisfy the hypotheses of 2.31 (with $T = j$). Hence,

$$g(x) = \lim_{n\to\infty} g_n(x)$$

exists for all x, is an element of $L_{\phi,\psi}$, and is such that

$$\lim_{n\to\infty} l_{\phi,\psi}(g - g_n) = 0.$$

Now, by 2.22 and 2.12,

$$\begin{aligned} l_{\phi,\psi}(g - g_n) &\leqq l_{\phi,\psi}(g - g_m) + l_{\phi,\psi}(g_m - g_n) \\ &\leqq l_{\phi,\psi}(g - g_n) + l_{\phi,\psi}(f_m - f_n) \\ &\leqq l_{\phi,\psi}(g - g_m) + 2^{-2n-2}, \text{ for all } m \geqq n. \end{aligned}$$

Thus,

$$l_{\phi,\psi}(g - g_n) \leqq 2^{-2n-2}, \qquad (n = 1, 2, \cdots).$$

This last estimate is of importance in that $M = 2^{-2n-2}$ is independent of j. As $j \to \infty$, we see that $g(x)$ converges almost everywhere (since E_j contracts and since meas $E_j \to 0$) to a limiting function which we shall denote by $f(x)$. Moreover, since for each n, $(n = 1, 2, \cdots)$,

$$g(x) - g_n(x) = \begin{cases} f(x) - f_n(x), & \text{if, simultaneously, } x \text{ is in } \bar{E}_j \text{ and } |x| \leqq j, \\ 0, & \text{otherwise,} \end{cases}$$

we find that if we fix n and allow $j \to \infty$, then the non-negative functions $|g(x) - g_n(x)|$ form a non-decreasing sequence converging almost everywhere to $|f(x) - f_n(x)|$. By 2.41, therefore,

$$l_{\phi,\psi}(f - f_n) \leqq 2^{-2n-2}, \qquad (n = 1, 2, \cdots).$$

This proves that $f(x)$ is the desired element of $L_{\phi,\psi}$.

III. Conditions for the Separability of $L_{\phi,\psi}$.

3.1. The property S.

Definition 3.11. $L_{\phi,\psi}$ will be said to have the property S if the corresponding functions $\phi(u)$, $\psi(u)$ satisfy the following additional requirements:

(3.111) $\phi(0+) = 0.$

(3.112) $0 < \psi(u) < \infty$ for all $u, 0 < u < \infty.$

(3.113)[5] There exists a constant $C, 1 \leqq C < \infty,$

such that

$$\phi(2u) \leqq C\,\phi(u), \qquad (0 < u < \infty).$$

THEOREM 3.12. *If $L_{\phi,\psi}$ satisfies* (3.113), *then*

$$0 < \phi(u) < \infty \quad \textit{whenever } 0 < u < \infty.$$

Proof. Since $\phi(1) = 1$, (3.113) yields:

$$0 < \phi(2^{-n}), \quad \phi(2^n) < \infty, \qquad (n = 0, 1, 2, \cdots).$$

Since $\phi(u)$ is non-decreasing, the theorem follows.

3.2. Dominated convergence in an $L_{\phi,\psi}$ having the property S.

THEOREM 3.21. *If $f(x)$ is an element of a space $L_{\phi,\psi}$ having the property S, then $|f(x)| < \infty$, almost everywhere.*

Proof. Apply (3.112) and 3.12.

THEOREM 3.22. *If $g(x)$, $f_1(x)$, $f_2(x), \cdots$ are elements of an $L_{\phi,\psi}$ with the property S, and if, almost everywhere,*

$$\begin{cases} f(x) = \lim\limits_{n\to\infty} f_n(x) \text{ exists}, \\ |f_n(x)| \leqq |g(x)|, \end{cases} \qquad (n = 1, 2, \cdots),$$

then $f(x)$ is an element of $L_{\phi,\psi}$, and

$$\lim_{n\to\infty} l_{\phi,\psi}(f - f_n) = 0.$$

Proof. We may assume that

$$l = l_{\phi,\psi}(g) > 0,$$

as otherwise the theorem is trivial. Let $\epsilon > 0$ be given. Let us determine a corresponding non-negative integer $m = m_\epsilon$ such that

$$2^m > \frac{4l}{\epsilon}.$$

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[5] Compare this requirement that $\phi(u)$ be "not too convex" with the property (Δ_2) used by Birnbaum and Orlicz: "Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potentzen," *Studia Mathematica*, t. 3 (1931), pp. 1-67.

Now $f(x)$, itself, is dominated almost everywhere by $g(x)$. Hence, by (3.113),

$$\phi\left(\frac{|f(x)-f_n(x)|}{\epsilon}\right) \leqq \phi\left(\frac{2|g(x)|}{\epsilon}\right) \leqq C^m \phi\left(\frac{2|g(x)|}{2^m\epsilon}\right) \leqq C^m \phi\left(\frac{|g(x)|}{2l}\right),$$

almost everywhere. But, by 2.11 and (3.112),

$$\int C^m \phi\left(\frac{|g(x)|}{2l}\right) dx \leqq C^m \psi(2l) < \infty.$$

Furthermore, by (3.111),

$$\lim_{n\to\infty} \phi\left(\frac{|f(x)-f_n(x)|}{\epsilon}\right) = 0, \text{ almost everywhere.}$$

By the Lebesgue theorem on dominated convergence, therefore,

$$\lim_{n\to\infty} \int \phi\left(\frac{|f(x)-f_n(x)|}{\epsilon}\right) dx = 0 < \psi(\epsilon).$$

In other words,

$$\limsup_{n\to\infty} l_{\phi,\psi}(f-f_n) \leqq \epsilon.$$

This being true for each $\epsilon > 0$, our theorem is established.

Note 3.23. If we apply 3.22 by making the substitutions:

$$g(x) = h(x), \quad f_n(x) = h_j(x), \qquad (n = j = 1, 2, \cdots),$$

we learn, under the hypotheses of 2.41 and the additional requirement that $L_{\phi,\psi}$ have the property S, that

$$\lim_{j\to\infty} l_{\phi,\psi}(h-h_j) = 0.$$

Now let $L_{\phi,\psi}$ fail to have the property S. If we can show that the above extension of 2.41 is here false, then we shall have shown at the same time that 3.22 would be false without the hypothesis that $L_{\phi,\psi}$ have the property S. Consider the one-parameter family

$$\{f_\lambda(x)\}, \qquad (0 \leqq \lambda \leqq 1),$$

which will be constructed in 3.41. Select, for example, the sequence,

$$\{1-1/j\}, \qquad (j = 1, 2, \cdots),$$

of parametric values and denote the corresponding elements of the family by $h_1(x)$, $h_2(x), \cdots$. These functions satisfy the hypotheses of 2.41 and converge for all x to the limiting function:

$$h(x) = \lim_{j\to\infty} h_j(x) = f_1(x).$$

But, by 3.41,

$$\liminf_{j\to\infty} l_{\phi,\psi}(h-h_j) \geqq l > 0.$$

Thus, when 3.41 will have received its independent proof, the present necessity for the property S will have been established.

THEOREM 3.24. *If $f(x)$ is an element of an $L_{\phi,\psi}$ having the property S, and if for each T, $(T=1,2,\cdots)$, we set*

$$f^{(T)}(x) = \begin{cases} f(x), & \text{if, simultaneously, } |f(x)| \leqq T \text{ and } |x| \leqq T, \\ 0, & \text{otherwise,} \end{cases}$$

then each $f^{(T)}(x)$ is an element of $L_{\phi,\psi}$, and

$$\lim_{T\to\infty} l_{\phi,\psi}(f-f^{(T)}) = 0.$$

Proof. By 2.12, each $f^{(T)}(x)$ is in $L_{\phi,\psi}$. By 3.21,

$$\lim_{T\to\infty} f^{(T)}(x) = f(x), \text{ almost everywhere.}$$

By setting $g(x)=f(x)$, we obtain the present theorem from 3.22.

THEOREM 3.25. *If $f(x)$ is measurable, is bounded, and differs from zero at most on a finite interval, and if $L_{\phi,\psi}$ has the property S, then $f(x)$ is an element of $L_{\phi,\psi}$.*

Proof. Apply (3.112) and 3.12.

3.3. Proof that each $L_{\phi,\psi}$ with the property S is separable.

Definition 3.31. The expression "rational step-function" will be reserved for functions of the form:

$$s(x) = \begin{cases} a_k, & |x-k\delta| < \delta/2, \quad (k=0,\pm 1,\cdots,\pm n), \\ 0, & \text{otherwise,} \end{cases}$$

where n is a positive integer, where δ is a positive rational number, and where each a_k is a complex number whose real and imaginary parts are both rational. It is to be noted that the totality of rational step-functions can be arranged in a sequence:

$$s_1(x), \qquad s_2(x), \cdots.$$

THEOREM 3.32. *If $f(x)$ is an element of a given $L_{\phi,\psi}$ having the property S, and if $\epsilon > 0$, then there can be selected from the sequence of rational step-functions an element, $s_j(x)$, such that*

$$l_{\phi,\psi}(f-s_j) < \epsilon.$$

Proof. In virtue of 2.22 and 3.24, we need prove the theorem only in that case in which there exists a positive integer T such that

$$\begin{cases} |f(x)| \leqq T, \text{ for all } x, \\ f(x) = 0, \text{ if } |x| > T. \end{cases}$$

Since $f(x)$ is also measurable, it is integrable. For each positive integer n let us determine the numbers:

$$b_{k,n} = n/T \int_{(k-\frac{1}{2})T/n}^{(k+\frac{1}{2})T/n} f(t)\, dt, \quad (k = 0, \pm 1, \cdots, \pm n).$$

Let $\{a_{k,n}\}$ be any corresponding rational complex numbers such that

$$|a_{k,n}| \leqq |b_{k,n}| \quad \text{and} \quad |a_{k,n} - b_{k,n}| < 1/n.$$

Consider the rational step-functions:

$$\sigma_n(x) = \begin{cases} a_{k,n}, & |x - kT/n| < T/2n, \quad (k = 0, \pm 1, \cdots, \pm n), \\ 0, & \text{otherwise}, \end{cases} \qquad (n = 1, 2, \cdots).$$

Almost everywhere, the derivative of the indefinite integral of $f(x)$ exists and is equal to $f(x)$. Whence, readily,

$$\lim_{n\to\infty} \sigma_n(x) = f(x), \text{ almost everywhere.}$$

Next, we observe that

$$|\sigma_n(x)| \leqq g(x) = \begin{cases} T, & |x| \leqq T, \\ 0, & \text{otherwise}, \end{cases} \qquad (n = 1, 2, \cdots).$$

By 3.25, $L_{\phi,\psi}$ contains $g(x)$ and each $\sigma_n(x)$. Therefore, by 3.22,

$$\lim_{n\to\infty} l_{\phi,\psi}(f - \sigma_n) = 0.$$

This fact, since $\{\sigma_n(x)\}$ is a sub-sequence of $\{s_j(x)\}$, establishes the theorem.

3.4. Spaces $L_{\phi,\psi}$ lacking the property S.

THEOREM 3.41. *If $L_{\phi,\psi}$ fails to have the property S, then there exist numbers l and M, $0 < l < M < \infty$, and there exists a one-parameter family $\{f_\lambda(x)\}$, $(0 \leqq \lambda \leqq 1)$, of elements of $L_{\phi,\psi}$ such that*

$$l_{\phi,\psi}(f_\lambda) \leqq M, \qquad l_{\phi,\psi}(f_\lambda - f_\mu) \geqq l, \qquad (\lambda \neq \mu).$$

In particular, there exist such elements for which, furthermore,

$$0 \leqq f_\lambda(x) \leqq f_\mu(x), \text{ for all } x, \qquad (\lambda < \mu).$$

Proof. Since, by 2.23, $l_{\phi,\psi}(f_\lambda - f_\mu) = l_{\phi,\psi}(f_\mu - f_\lambda)$, we shall introduce no loss of generality if we replace the requirement $\lambda \neq \mu$ by $\lambda < \mu$. Throughout the following four cases it will be agreed, firstly, that each definition or relation involving λ alone holds for all λ such that $0 \leqq \lambda \leqq 1$, and secondly, that each inequality involving both λ and μ holds whenever $0 \leqq \lambda < \mu \leqq 1$.

Case A. (3.111) is false.

Here, by 1.21, we can determine $l, 0 < l < 1$, so that

$$0 \leqq \psi(l) < \psi(1) < \infty.$$

And we can find a corresponding $M > 1$ such that

$$\phi\left(\frac{1}{M}\right)\psi(l) < \phi(0+)\psi(1).$$

Now, let

$$f_\lambda(x) = \begin{cases} \lambda, & 0 < x < \dfrac{\psi(1)}{\phi\left(\dfrac{1}{M}\right)}, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\int \phi\left(\frac{f_\lambda(x)}{M}\right) dx \leqq \psi(1) \leqq \psi(M),$$

we see that

$$l_{\phi,\psi}(f_\lambda) \leqq M.$$

And since

$$\int \phi\left(\frac{f_\mu(x) - f_\lambda(x)}{l}\right) dx \geqq \phi(0+)\,\frac{\psi(1)}{\phi\left(\dfrac{1}{M}\right)} > \psi(l),$$

we conclude:

$$l_{\phi,\psi}(f_\mu - f_\lambda) \geqq l.$$

Case B. (3.113) is false.

Here, there exists a sequence: $u_1, u_2, \cdots, 0 < u_k < \infty$, such that

$$\phi(2u_k) > k\,\phi(u_k), \qquad (k = 1, 2, \cdots).$$

Determine corresponding positive numbers $c_1, c_2, \cdots$ so that

$$\sum_{k=1}^{\infty} c_k\,\phi(u_k) \leqq \psi(1), \qquad \sum_{k=1}^{\infty} c_k\,\phi(2u_k) = \infty.$$

We assert that the constants $l = 1$ and $M = 2$, and the functions:

$$f_\lambda(x) = \begin{cases} 2u_k, & 0 < x - \sum_{n=1}^{k-1} c_n < \lambda c_k, \qquad (k = 1, 2, \cdots), \\ 0, & \text{otherwise,} \end{cases}$$

meet the requirements of the theorem. Since

$$\int \phi\left(\frac{f_\lambda(x)}{2}\right) dx = \lambda \sum_{k=1}^{\infty} c_k \phi(u_k) \leqq \psi(1) \leqq \psi(2),$$

we see that

$$l_{\phi,\psi}(f_\lambda) \leqq 2.$$

And since

$$\phi(f_\mu(x) - f_\lambda(x))\, dx = \sum_{k=1}^{\infty} (\mu - \lambda) c_k \phi(2u_k) = \infty > \psi(1),$$

we conclude:

$$l_{\phi,\psi}(f_\mu - f_\lambda) \geqq 1.$$

Case C. (3.113) is valid; (3.112) is false because there exists $l, 0 < l < 1$, such that $\psi(l) = 0$.

For this l and for $M = 1$, we assert that the functions:

$$f_\lambda(x) = \begin{cases} \lambda, & 0 < x < \psi(1), \\ 0, & \text{otherwise,} \end{cases}$$

meet the requirements of the theorem. Since $\phi(1) = 1$, we see that

$$\int \phi(f_\lambda(x))\, dx \leqq \psi(1),$$

and hence that

$$l_{\phi,\psi}(f_\lambda) \leqq 1.$$

Since 3.12 yields:

$$\int \phi\left(\frac{f_\mu(x) - f_\lambda(x)}{l}\right) dx > 0 = \psi(l),$$

we conclude:

$$l_{\phi,\psi}(f_\mu - f_\lambda) \geqq l.$$

Case D. (3.112) is false because there exists M, $1 < M < \infty$, such that $\psi(M) = \infty$.

For $l = 1$ and for the above M, we assert that the functions:

$$f_\lambda(x) = \begin{cases} 1, & 0 < x - k < \lambda, \qquad (k = 1, 2, \cdots), \\ 0, & \text{otherwise,} \end{cases}$$

meet the requirements of the theorem. Since $\psi(M) = \infty$, we see that

$$l_{\phi,\psi}(f_\lambda) \leqq M.$$

And since

$$\phi(f_\mu(x) - f_\lambda(x))\,dx = \sum_{k=1}^{\infty} (\mu - \lambda) = \infty > \psi(1),$$

we conclude:

$$l_{\phi,\psi}(f_\mu - f_\lambda) \geqq 1.$$

THEOREM 3.42. *If $L_{\phi,\psi}$ is separable, then $L_{\phi,\psi}$ has the property S.*

Proof. Assume that $L_{\phi,\psi}$ lacks the property S, but that there exists a sequence $g_1(x), g_2(x), \cdots$ dense in $L_{\phi,\psi}$. Consider the constant l and the family $\{f_\lambda(x)\}$ furnished by 3.41. For each $f_\lambda(x)$, determine a corresponding $g_j(x)$ such that

$$l_{\phi,\psi}(f_\lambda - g_j) < l/3,$$

and write: $f_\lambda \sim g_j$. Now, by 2.22,

$$l \leqq l_{\phi,\psi}(f_\mu - f_\lambda) \leqq 2l/3 + l_{\phi,\psi}(g_k - g_j),$$

whenever $f_\lambda \sim g_j$ and $f_\mu \sim g_k$. Thus, by 2.21, the correspondence between the non-denumerable family and the sequence is one-to-one. This contradiction establishes the theorem.

COLUMBIA UNIVERSITY.

THE ISOPERIMETRIC INEQUALITY ON THE SPHERE.[1]

By TIBOR RADÓ.

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1. *Statement and preliminary discussion of results. Reduction to a geometrical lemma.*

1.1. Among all simple closed plane curves with given length the circle has the greatest area. This classical theorem can also be expressed as follows: The area a of the circle with perimeter l is given by the equation

$$4\pi a = l^2.$$

Hence, on account of the extremal property of the circle, we have for every simple closed plane curve with length l and enclosed area a the *isoperimetric inequality in the plane*

$$4\pi a \leqq l^2,$$

where the sign of equality holds if and only if the curve is a circle.

1.2. Similar considerations, applied to simple closed curves on the unit sphere, lead us to surmise the following *isoperimetric inequality on the unit sphere*: if l is the length and a the area enclosed by a simple closed curve on the unit sphere, then

$$4\pi a - a^2 \leqq l^2, \tag{1}$$

where the sign of equality holds if and only if the curve is a circle.

Since a simple closed curve determines *two* regions on the unit sphere, with areas a and $4\pi - a$ respectively, it is interesting to observe that the inequality (1) holds for *both* of these regions. Indeed, the function $\phi(x) = 4\pi x - x^2$ satisfies the equation

$$\phi(x) = \phi(4\pi - x),$$

and hence, if (1) holds for one of the two regions determined by a curve, then (1) also holds for the other region.

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[1] Presented to the American Mathematical Society at the meeting in Chicago, April, 1935.

1. 3. The inequality (1) has been proved by F. Bernstein for the important special case of curves which can be covered by a hemisphere,[2] and as far as the present author is aware, the general case has not been investigated as yet. It is the purpose of the present paper to establish (1) in complete generality.

1. 4. In the sequel, C will denote a simple closed curve on the unit sphere, l the length of C, and a the area of any one of the two regions determined by C on the unit sphere. The following two cases exhaust then all possibilities.

Case I. The curve C can be covered by some open hemisphere.[3]

Case II. There exists no open hemisphere covering C.

1. 5. Case I is settled by the result of F. Bernstein (see 1. 3). While it seems that the ingenious method of F. Bernstein does not apply to case II, we shall see presently that this case is easily disposed of by means of the following lemma.

LEMMA. *If a simple closed curve C, located on the unit sphere, has a length less than 2π, then C can be covered by some open hemisphere.*

1. 6. We shall prove the above lemma in § 2. If this result is anticipated, the discussion of case II in 1. 4 is quite trivial. Indeed, if C cannot be covered by any open hemisphere, then the length l of C is $\geqq 2\pi$, on account of the lemma. On the other hand, the maximum of the function

$$\phi(x) = 4\pi x - x^2$$

is

$$\phi(2\pi) = (2\pi)^2.$$

Hence (1) is certainly satisfied if $l \geqq 2\pi$, and the sign of equality can hold if and only if $l = 2\pi$, $a = 2\pi$. But then C divides the surface of the unit sphere into two regions with equal areas, and consequently C contains two points P_1, P_2 which are end-points of the same diameter of the unit sphere.[4] From $l = 2\pi$, in connection with the fact that the geodesics on the unit sphere are

[2] *Mathematische Annalen*, vol. 60 (1905), pp. 117-136.

[3] An open hemisphere is a hemisphere *without* its boundary circle, and a closed hemisphere is a hemisphere *with* its boundary circle.

[4] *Loc. cit.* [2].

arcs of great circles, it follows then that C consists of two arcs of great circles joining P_1 and P_2. On account of $a = 2\pi$ it follows finally that these arcs are sub-arcs of the same great circle; that is to say, that C itself is a great circle.

We proceed now to prove the lemma of 1. 5.

2. *Proof of the geometrical lemma.*

2. 1. In the sequel, C denotes a simple closed curve on the unit sphere, $l(C)$ denotes the length of C, *and it is assumed that* $l(C) < 2\pi$.

2. 2. Obviously, C *cannot contain two points which are end-points of the same diameter of the unit sphere.* Indeed, if P_1, P_2 were two such points, then both arcs determined on C by P_1, P_2 would have a length $\geqq \pi$ and thus we should have $l(C) \geqq 2\pi$, while by assumption $l(C) < 2\pi$.

2. 3. Let A_1, A_2, A_3 be three points of intersection of a great circle Γ with C. These points determine three non-overlapping arcs Γ_1, Γ_2, Γ_3 on Γ and three non-overlapping arcs C_1, C_2, C_3 on C, where Γ_1, C_1 are the arcs bounded by A_2, A_3, and so on. *Then one of the arcs* Γ_1, Γ_2, Γ_3 *has a length* $> \pi$. Indeed, if we assume that

$$l(\Gamma_1) \leqq \pi,\ l(\Gamma_2) \leqq \pi,\ l(\Gamma_3) \leqq \pi,$$

then Γ_1, Γ_2, Γ_3 are all three shortest arcs, on the unit sphere, between their respective end-points. Hence

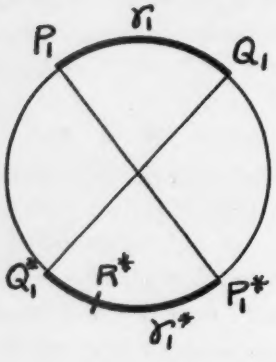
$$l(C) = l(C_1) + l(C_2) + l(C_3) \geqq l(\Gamma_1) + l(\Gamma_2) + l(\Gamma_3) = 2\pi,$$

in contradiction with the assumption that $l(C) < 2\pi$.

2. 4. Denote by S the set of all the points of intersection of a great circle Γ with C. In case S is empty, the conclusion to be stated at the end of the present section is obvious. Therefore we assume that S is not empty. Then S is a closed set, and the complementary set on Γ is the sum of non-overlapping open arcs γ_1, $\gamma_2, \cdots$. Since the sum of the lengths of these complementary arcs is finite, namely $\leqq 2\pi$, there is clearly one among them whose length is greater than or equal to the lengths of all the others. Suppose the notation is such that

$$l(\gamma_1) \geqq l(\gamma_n), \qquad (n = 1, 2, 3, \cdots).$$

Denote by P_1, Q_1 the end-points of γ_1. Then P_1, Q_1 are points of S and consequently of C, and hence (see 2. 2) P_1 and Q_1 are not end-points of the same diameter of Γ. Consequently, $l(\gamma_1) \neq \pi$. We wish to show that $l(\gamma_1) > \pi$. Suppose, indeed, that



$$(2) \qquad l(\gamma_1) < \pi.$$

Denote by P_1^*, Q_1^* the points of Γ diametrically opposite to P_1, Q_1, respectively, and by γ_1^* the sub-arc of Γ which is bounded by P_1^*, Q_1^*, and which does not overlap with γ_1. On account of the assumption (2), such a sub-arc γ_1^* does exist (see figure) and we have

$$l(\gamma_1^*) = l(\gamma_1).$$

On account of 2. 2, the points P_1^*, Q_1^* are not in the set S. Since S is closed, we have therefore vicinities of P_1^* and Q_1^* which are free of points of S. Hence, if we first assume that γ_1^* does not contain any point of S, then there follows the existence of an open sub-arc $\bar{\gamma}$ of Γ which is free of points of S and which is *longer* than γ_1^*. Such an arc $\bar{\gamma}$ would then be comprised in one of the complementary arcs $\gamma_1, \gamma_2, \cdots$, say in γ_k, and we should have

$$l(\gamma_k) \geqq l(\bar{\gamma}) > l(\gamma_1^*) = l(\gamma_1).$$

This contradicts however the extremal property of γ_1. Secondly, let us assume that γ_1^* contains some point R^* of S. Then R^* is different both from P_1^* and Q_1^*, since these points are not in S. Hence (see figure) none of the three non-overlapping arcs, determined by P_1, Q_1, R^* on Γ, has a length $> \pi$. This contradicts however 2. 3. Summing up:

On every great circle Γ there exists an open sub-arc with a length $> \pi$ which is free of points of C.

2. 5. It will be convenient to use the following definitions. A point-set Σ, located on the unit sphere, will be said to possess the property *CH* if it can be covered by some closed hemisphere. The set Σ will be said to possess the property *OH* if it can be covered by some open hemisphere.

2. 6. *Let Σ be a closed sub-arc of the curve C* (the case $\Sigma \equiv C$ is not excluded). *If Σ possesses the property CH, then it also possesses the property OH.*

For the sake of brevity of presentation, the closed hemisphere which covers Σ will be called the southern hemisphere, and the boundary of this hemisphere will be called the equator E. On account of 2. 4, we have then on the equator E an open sub-arc $\bar{\gamma}$, free of points of C and consequently of points of Σ, such that $l(\bar{\gamma}) > \pi$. We can choose then on $\bar{\gamma}$ a closed sub-arc γ such that $l(\gamma) = \pi$. Denote by A, B the end-points of γ. Then A and B are the end-points of the same diameter of the unit sphere. Since γ and Σ are closed sets without common points, a slight rotation around the diameter AB will carry γ into an arc γ^* which has no points in common with Σ. Let us denote by E^* the transform of the equator E under such a rotation. Let us choose the sense of rotation in such a way that γ^* is located on the closed southern hemisphere. Then the open arc $E^* - \gamma^*$ is located on the *open* northern hemisphere, and has consequently no points in common with Σ which is located on the *closed* southern hemisphere. Since γ^* also has no common points with Σ, it follows that the great circle $E^* = \gamma^* + (E^* - \gamma^*)$ does not intersect the arc Σ. Hence Σ is located on one of the two open hemispheres bounded by E^*, that is to say Σ has the property OH.

2. 7. Let us now choose a sense on the curve C which we shall call the positive sense, and let us also choose a point A on C. We shall denote by $P(s)$ the point of C at distance s from A, the distance being measured on C in the positive sense from A to $P(s)$. Then s varies in the closed interval

$$0 \leqq s \leqq l(C) < 2\pi. \tag{3}$$

2. 8. Let us denote by $\{s; CH\}$ the set of all those values of s, in the interval (3), for which the following assertion is true: the closed sub-arc $\gamma(s)$ of C, consisting of all points $P(\sigma)$ such that $0 \leqq \sigma \leqq s$, possesses the property CH. The set $\{s; CH\}$ is not empty, since sufficiently small values of s are certainly comprised in $\{s; CH\}$. Furthermore, $\{s; CH\}$ is obviously a closed set, and contains therefore a largest number $\bar{s}$. We assert that $\bar{s} = l(C)$. Suppose, indeed, that $\bar{s} < l(C)$. Then the arc $\gamma(\bar{s})$ is a true sub-arc of C. Since $\bar{s}$ is in $\{s; CH\}$, the arc $\gamma(\bar{s})$ possesses the property CH, and therefore, on account of 2. 6, the property OH also. That is to say, the closed arc $\gamma(\bar{s})$ can be covered by some open hemisphere. By obvious reasons of continuity, the same open hemisphere will also cover the arc $\gamma(\bar{s} + \epsilon)$, provided $\epsilon > 0$ is sufficiently small. Hence, for $\epsilon > 0$ sufficiently small, the arc $\gamma(\bar{s} + \epsilon)$ possesses the property OH and consequently the property CH. This contradicts however the extremal property of $\bar{s}$. Hence $\bar{s} = l(C)$, that is to say the whole curve C possesses the property CH and therefore, on account of 2. 6 the property OH also. Thus the lemma of 1. 5 is proved.

Conclusion. The following general problem arises naturally in connection with the topics considered in the present paper. Let there be given a general surface S, subject only to the usual restrictions of regularity as made in differential geometry. Denote by a the area and by l the length of the perimeter of a simply connected region on S. What is the best limitation, valid for all such regions on S, for a in terms of l? The reader might welcome a few brief references indicating the scope of the results obtained up to the present.

If the surface S is a plane, then the classical isoperimetric inequality $4\pi a \leqq l^2$ solves the problem. If S is the unit sphere, then the answer is $4\pi a - a^2 \leqq l^2$, as discussed in the present paper. If S is a minimal surface, then Carleman proved that the answer is $4\pi a \leqq l^2$, the same as for the plane (T. Carleman, "Zur Theorie der Minimalflächen," *Mathematische Zeitschrift*, vol. 9 (1921), pp. 154-160). If S is a saddle-shaped surface, that is to say if the Gaussian curvature of S is $\leqq 0$ throughout, then the answer is again $4\pi a \leqq l^2$. And conversely, if the inequality $4\pi a \leqq l^2$ holds for all regions on a surface, S, then the Gaussian curvature of S is necessarily $\leqq 0$ (these last two results are due to E. F. Beckenbach and T. Radó, "Subharmonic functions and surfaces of negative curvature," *Transactions of the American Mathematical Society*, vol. 35, pp. 662-674). Obviously a number of attractive problems of this nature are yet to be solved.

THE OHIO STATE UNIVERSITY,
COLUMBUS, OHIO.

THE CONJUGATE CHORD QUADRICS OF A CURVE ON A SURFACE.

By M. L. MacQueen.

1. *Introduction.* The projective differential geometry of a surface in ordinary space has been greatly enriched by the knowledge of a number of quadrics that have been associated with a point of a surface. Among these quadrics there are the two *asymptotic osculating quadrics* of Bompiani;[1] each of these is the limit of the quadric determined by three asymptotic tangents of one family constructed at points of a curve C on a surface S at these points independently approach P along C.

Lane[2] has defined in a similar way two quadrics called *conjugate osculating quadrics,* replacing, however, in the definition just stated, the three asymptotic tangents by three consecutive tangents of the curves of one family of a given conjugate net.

Another pair of quadrics called *asymptotic chord quadrics* has been defined[3] by Bompiani, these quadrics having a geometrical property in common with the asymptotic osculating quadrics, namely, that they contain the asymptotic tangents at the point P on the surface S. An asymptotic chord quadric may be defined as follows: On a surface S let us select a curve C and two points P and $\bar{P}$ on this curve. Denote the points of intersection of the two asymptotic curves through each of these points by P_1 and P_2. As $\bar{P}$ approaches P along C the asymptotic chord $\bar{P}P_1$ generates a ruled surface which has for every position of its generators a well defined osculating quadric. The limiting position of this quadric as $\bar{P} \to P$ along C is called an asymptotic chord quadric. A second quadric may be defined similarly by considering the asymptotic chord $\bar{P}P_2$.

It would seem to be of interest to define similarly two quadrics called *conjugate chord quadrics,* which are associated with each point of a given

[1] E. Bompiani, "Ancora sulla geometria delle superficie considerate nello spazio rigato," *Rendiconti dei Lincei* (6), vol. 4 (1926), p. 262.

[2] E. P. Lane, "Conjugate nets and the lines of curvature," *American Journal of Mathematics,* vol. 53 (1931), pp. 573-588. Hereinafter referred to as Lane, "Conjugate Nets."

[3] E. Bompiani, "Sugli elementi di 2° ordine delle curve di una superficie," *Rendiconti dei Lincei* (6), vol. 9 (1929), p. 288.

curve on a surface sustaining a given conjugate net. For this purpose we shall replace the asymptotic curves through P and $\bar{P}$, in the definition of an asymptotic chord quadric, by the curves of each family of a given conjugate net.

In § 2 we summarize briefly for subsequent use some of the theory of the projective differential geometry of a surface in ordinary space referred to a conjugate net as parametric. In § 3, after formulating a definition of the conjugate chord quadrics, the equations of these quadrics are found. The method which we employ is quite similar to that used by Bompiani in obtaining the equations of the asymptotic chord quadrics. Finally, in § 4, we study briefly the geometrical properties of these quadrics, our investigation paralleling somewhat that of the conjugate osculating quadrics made by Lane.

2. *Analytic basis.* In this section an analytic basis for the projective differential geometry of a parametric conjugate net on a surface in ordinary space is established. We employ the completely integrable system of differential equations, which are written in a symmetrical form intimately associated with the axis congruence of the net, that define a conjugate net in ordinary space, except for a projective transformation.

Let the projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$ of a point P_x on a surface S referred to a conjugate net N_x in ordinary space be given as analytic functions of two independent variables u, v. The two osculating planes of the parametric curves C_u, C_v at a point P_x on the surface S intersect in a line defined to be the axis of P_x with respect to the net N_x. If P_y is any point distinct from P_x on the axis and if P_y is chosen such that it is the harmonic conjugate of P_x with respect to the two foci of the axis regarded as generating a congruence when the point P_x varies over the surface S, then x and y satisfy a system of differential equations [4] of the form

$$
\begin{aligned}
x_{uu} &= px + \alpha x_u + Ly, \\
x_{uv} &= cx + ax_u + bx_v, \\
x_{vv} &= qx + \delta x_v + Ny \qquad (LN \neq 0),
\end{aligned} \tag{1}
$$

where the notation here employed is similar to that used by Lane in his recent book.

The coefficients of system (1) are functions of u, v and satisfy four integrability conditions which we do not need to write here.

From the equations

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[4] E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932, p. 138.

$$(x_{vv})_u = (x_{uv})_v, \qquad (x_{uu})_v = (x_{uv})_u$$

we find

$$\begin{aligned} y_u &= fx - nx_u + sx_v + Ay, \\ y_v &= gx + tx_u + nx_v + By, \end{aligned} \tag{2}$$

where we have placed

$$\begin{aligned} fN &= c_v + ac + bq - c\delta - q_u, & gL &= c_u + bc + ap - c\alpha - p_v, \\ -nN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v, \\ sN &= b_v + ab + c - \delta_u, & nL &= b_u + b^2 - b\alpha - p, \\ A &= b - (\log N)_u, & B &= a - (\log L)_v. \end{aligned} \tag{3}$$

The Laplace-Darboux point invariants H, K, the tangential invariants $\boldsymbol{H}$, $\boldsymbol{K}$, the invariants $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}$ of Green, and the invariant r are given in the notation of system (1) by the formulas

$$\begin{aligned} H &= c + ab - a_u, & K &= c + ab - b_v, \\ \boldsymbol{H} &= sN, & \boldsymbol{K} &= tL, \\ 8\mathfrak{B}' &= 4a - 2\delta + (\log r)_v, & 8\mathfrak{C}' &= 4b - 2\alpha - (\log r)_u, \\ \mathfrak{D} &= -2nL, & r &= N/L. \end{aligned} \tag{4}$$

The differential equation of the asymptotic curves on the surface S is

$$L\,du^2 + N\,dv^2 = 0. \tag{5}$$

Since the four points x, x_u, x_v, y are not coplanar, we shall choose them as vertices of a local tetrahedron of reference at P_x. Let the unit point be chosen so that a point X defined by

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 y \tag{6}$$

has local coördinates proportional to $x_1, \cdots, x_4$. Later we shall have occasion to use the covariant tetrahedron with vertices at the points x, ρ, σ, y for which

$$\sigma = x_v - ax, \qquad \rho = x_u - bx. \tag{7}$$

The points σ, ρ are the ray-points, or Laplace transformed points, of the curves C_u, C_v respectively, corresponding to the point P_x.

3. *The conjugate chord quadrics.* In order to formulate a definition let us consider a surface S referred to a conjugate net N_x, and any curve C on S not belonging to N_x. Let P_x and $\bar{P}$ be two distinct points of the curve C. Let $P_1(P_2)$ be the point of intersection of the u-curve (v-curve) through P_x and the v-curve (u-curve) through $\bar{P}$. As the point $\bar{P}$ approaches P_x along C,

the line joining $\bar{P}$ to P_1, namely, the conjugate chord v, describes a ruled surface which has for every position of one of its generators a well defined osculating quadric. The limit of this quadric as $\bar{P}$ approaches P_x along C is a *conjugate chord quadric* at the point P_x of the curve C on the net N_z. A second conjugate chord quadric is defined in a way similar to the foregoing by considering the conjugate chord $\bar{P}P_2$.

In order to find the equations of the conjugate chord quadrics just defined, let us consider a curve C through points P_x and $\bar{P}$ on S to be defined by the parametric equations

$$u = u(t), \qquad v = v(t). \tag{8}$$

If the points P_x and $\bar{P}$ of the curve C correspond to the parameter values $t = 0$ and $t = t$, and if the coördinates of points $\bar{P}$, P_1, and P_2 be denoted by $\bar{x}$, x_1, and x_2 respectively, we have

$$\begin{aligned} \bar{x} &= x + A_1 t + A_2 t^2/2 + A_3 t^3/\underline{3} + \cdots, \\ x_1 &= x + R_1 t + R_2 t^2/2 + R_3 t^3/\underline{3} + \cdots, \\ x_2 &= x + S_1 t + S_2 t^2/2 + S_3 t^3/\underline{3} + \cdots, \end{aligned} \tag{9}$$

where

$$\begin{aligned} A_1 &= x_u u' + x_v v', \qquad A_2 = x_{uu}u'^2 + 2x_{uv}u'v' + x_{vv}v'^2 + x_u u'' + x_v v'', \\ A_3 &= x_{uuu}u'^3 + 3x_{uuv}u'^2v' + 3x_{uvv}u'v'^2 + x_{vvv}v'^3 + 3x_{uu}u'u'' \\ &\quad + 3x_{uv}(u'v'' + u''v') + 3x_{vv}v'v'' + x_u u''' + x_v v''', \end{aligned} \tag{10}$$

and

$$\begin{aligned} R_1 &= x_u u', \quad R_2 = x_{uu}u'^2 + x_u u'', \quad R_3 = x_{uuu}u'^3 + 3x_{uu}u'u'' + x_u u''', \\ S_1 &= x_v v', \quad S_2 = x_{vv}v'^2 + x_v v'', \quad S_3 = x_{vvv}v'^3 + 3x_{vv}v'v'' + x_v v'''. \end{aligned} \tag{11}$$

The formulas for the third derivatives of x expressed as linear combinations of x, x_u, x_v, y are found from system (1) by differentiation and substitution to be

$$\begin{aligned} x_{uuu} &= (p_u + \alpha p + fL)x + (\alpha_u + \alpha^2 + p - nL)x_u \\ &\qquad + sLx_v + (L_u + \alpha L + AL)y, \\ x_{uuv} &= (c_u + bc + ap)x + (a_u + c + ab + a\alpha)x_u \\ &\qquad + (b_u + b^2)x_v + aLy, \\ x_{uvv} &= (c_v + ac + bq)x + (a_v + a^2)x_u \\ &\qquad + (b_v + c + ab + b\delta)x_v + bNy, \\ x_{vvv} &= (q_v + \delta q + gN)x + tNx_u \\ &\qquad + (\delta_v + \delta^2 + q + nN)x_v + (N_v + \delta N + BN)y. \end{aligned} \tag{12}$$

It is evident that the points X, Y defined by

$$X = (\bar{x} + x_1)/2, \qquad Y = (\bar{x} - x_1)/t \tag{13}$$

are on the conjugate chord $\bar{P}P_1$. Any point P_ϕ, except the point Y, on the line $\bar{P}P_1$ is defined by placing

$$\phi(t,\mu) = X(t) + \mu Y(t) \qquad (\mu \text{ scalar}). \tag{14}$$

If t is fixed while μ varies, the locus of the point P_ϕ is the line $\bar{P}P_1$. If t, μ both vary, the locus of P_ϕ is the ruled surface of the conjugate chords of one family (v-curves) relative to the curve C.

Fixing a generator of the ruled surface by holding t fixed, we observe that an asymptotic tangent at a point of the fixed generator of the ruled surface, distinct from a rectilinear generator, at the point (t,μ) joins the point P_ϕ to the point $d\phi$, if $dt/d\mu$ is determined by the condition

$$|\, \phi, \phi_\mu, \phi_t, \phi_{tt}dt + 2\phi_{t\mu}d\mu \,| = 0. \tag{15}$$

If we vary μ, with $dt/d\mu$ determined by (15), t being fixed, the line joining ϕ and $d\phi$ describes an osculating quadric along a fixed generator. When t approaches zero we obtain the desired quadric at P_x.

It is easy to verify that condition (15) may be written in the form

$$|\, X, Y, X' + \mu Y', \phi_{tt} \,|\, dt + 2 \,|\, X, Y, X', Y' \,|\, d\mu = 0, \tag{16}$$

wherein the accents denote derivatives with respect to t. When $t = 0$, easy calculations yield the following results:

$$\begin{aligned}
X &= x, \qquad Y = x_v v', \qquad X' = x_u u' + x_v v'/2, \\
Y' &= x_{uv}u'v' + x_{vv}v'^2/2 + x_v v''/2, \\
X'' &= x_{uu}u'^2 + x_{uv}u'v' + x_{vv}v'^2/2 + x_u u'' + x_v v''/2, \\
Y'' &= x_{uuv}u'^2v' + x_{uvv}u'v'^2 + x_{vvv}v'^3/3 + x_{uv}(u'v'' + u''v') \\
&\quad + x_{vv}v'v'' + x_v v'''/3.
\end{aligned} \tag{17}$$

On substituting in equation (16) the expression for ϕ_{tt} found from equation (14), and reducing by means of elementary properties of determinants, we find that the first determinant appearing in (16) may be written in the form

$$\begin{aligned}
|\, X, Y, X' + \mu Y', \phi_{tt} \,| &= |\, X, Y, X' + \mu Y', X'' + \mu Y'' \,| \\
&= D + E\mu + F\mu^2,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
D &= |\, X, Y, X', X'' \,|, \qquad F = |\, X, Y, Y', Y'' \,|, \\
E &= |\, X, Y, X', Y'' \,| + |\, X, Y, Y', X'' \,|.
\end{aligned} \tag{19}$$

On making use of (17), (1), and (12) we find

$$
\begin{aligned}
&|\,X, Y, X', Y'\,| = -N\Delta u'v'^3/2,\\
&D = -(Lu'^2 + Nv'^2/2)\Delta u'v',\\
&E = \{N(\alpha u'^2 + au'v' + u'')v'/2 - a(Lu'^2 + Nv'^2/2)u'\\
&\quad -[aLu'^2 + bNu'v' + (N_v + \delta N + BN)v'^2/3 + Nv'']u'\}\Delta v'^2,\\
&F = \{N[(a_u + c + ab + a\alpha)u'^2v' + (a_v + a^2)u'v'^2 + tNv'^3/3\\
&\quad + a(u'v'' + u''v')]/2\\
&\quad - au'[aLu'^2 + bNu'v' + (N_v + \delta N + BN)v'^2/3 + Nv'']\}\Delta v'^3,
\end{aligned}
\tag{20}
$$

where

$$\Delta = |\,x, x_u, x_v, y\,| \neq 0.$$

By means of the results just obtained it is now possible to write equation (16) in the form

$$(\bar{D} + \bar{E}\mu + \bar{F}\mu^2)dt = Nu'v'^2 d\mu, \tag{21}$$

where we have placed

$$\bar{D} = D/\Delta v', \qquad \bar{E} = E/\Delta v', \qquad \bar{F} = F/\Delta v'.$$

The quadric whose equation we are seeking is generated by the point

$$\zeta = k\phi + d\phi = k\phi + \phi_t dt + \phi_\mu d\mu \qquad (k \text{ scalar}), \tag{22}$$

as μ and k vary and if $dt/d\mu$ is determined by (21). On making use of (14), (17), and (21), we find that ζ can be expressed in the form

$$
\begin{aligned}
\zeta = {}& k(x + \mu x_v v') + Nu'v'^2[(x_u u' + x_v v'/2)\\
&+ \mu(x_{uv}u'v' + x_{vv}v'^2/2 + x_v v''/2)] + (\bar{D} + \bar{E}\mu + \bar{F}\mu^2)x_v v'.
\end{aligned}
\tag{23}
$$

Then by use of (1) we find

$$\zeta = x_1 x + x_2 x_u + x_3 x_v + x_4 y,$$

where

$$
\begin{aligned}
x_1 &= k + N\mu u'v'^3(cu' + qv'/2),\\
x_2 &= Nu'^2v'^2(1 + a\mu v'),\\
x_3 &= k\mu v' + Nu'v'^3/2 + N\mu u'v'^3\,(2bu' + \delta v' + v''/v')/2\\
&\quad + (\bar{D} + \bar{E}\mu + \bar{F}\mu^2)v',\\
x_4 &= N^2\mu u'v'^4/2.
\end{aligned}
\tag{24}
$$

These are the local coördinates $x_1, \cdots, x_4$ of the point P_ζ, referred to the tetrahedron x, x_u, x_v, y with suitably chosen unit point. Homogeneous elimination of μ and k from equations (24), together with some further simplification, the details of which will be omitted, gives the *equation of the conjugate chord quadric Q_v, referred to the tetrahedron x, x_u, x_v, y,* namely,

$$(25) \qquad L\lambda x_2^2 + N\lambda^2 x_2 x_3 - 2a\lambda x_3 x_4 - 2\lambda x_1 x_4 + P x_4^2 + Q x_2 x_4 = 0,$$

wherein we have placed $\lambda = dv/du$, and

$$P = 2[(c + ab - a_u) + (q + a\delta - a_v - a^2)\lambda - tN\lambda^2/3]/N,$$
$$Q = 2(N_v/N - \delta/2 + B)\lambda^2/3 - \alpha\lambda + \lambda(\log\lambda)'.$$

In order to write the equation of the quadric Q_v referred to the tetrahedron x, ρ, σ, y, it is not difficult to show that it is sufficient to replace x_1 in equation (25) by $x_1 - bx_2 - ax_3$. On making this substitution and simplifying the coefficients by means of (3) and (4) we find the *equation of the conjugate chord quadric Q_v referred to the covariant tetrahedron x, ρ, σ, y to be*

$$(26) \quad Lx_2^2 + [4(\mathfrak{C}' + \lambda\mathfrak{B}'/3) + (\log\lambda r^{1/2})']x_2x_4 - [\mathfrak{D}/L - 2H/\lambda N + 2\lambda\boldsymbol{K}/3L]x_4^2 - 2x_1x_4 + \lambda N x_2 x_3 = 0.$$

The equation of the conjugate chord quadric Q_u at the point P_x of the curve C on the net N_x can be written immediately by interchanging u and v and making the necessary symmetrical interchanges of the other symbols. For this result we find

$$(27) \quad N\lambda x_3^2 + [4(\mathfrak{C}'/3 + \lambda\mathfrak{B}') - \lambda(\log\lambda r^{1/2})']x_3x_4 + [\lambda\mathfrak{D}/L + 2\lambda^2 K/L - 2\boldsymbol{H}/3N]x_4^2 - 2\lambda x_1x_4 + Lx_2x_3 = 0.$$

4. *Geometrical considerations.* Let us regard the curve C, employed in defining the conjugate chord quadrics, as imbedded in the one-parameter family of curves defined on the surface S by the equation

$$(28) \qquad dv - \lambda du = 0,$$

where λ is a non-vanishing function of u, v. Then the differential equation

$$(29) \qquad (dv - \lambda du)(dv - \mu du) = 0$$

will represent a conjugate net on the surface S if the two directions defined by this equation separate harmonically the two asymptotic directions satisfying equation (5). A necessary and sufficient condition for this is

$$(30) \qquad \mu = -1/\lambda r.$$

The two curves of such a conjugate net that pass through a point on the surface S will be denoted by C_λ and C_μ respectively, according as the direction dv/du has the value λ or μ.

The tangent line of the curve C_λ at the point P_x is given by

$$x_4 = x_3 - \lambda x_2 = 0, \tag{31}$$

and that of the curve C_μ is

$$x_4 = x_2 + \lambda r x_3 = 0. \tag{32}$$

Inspection of equation (27) shows that the conjugate chord quadric Q_u is intersected by the tangent plane, $x_4 = 0$, of the surface S in the u-tangent, $x_4 = x_3 = 0$, and in a residual line which is precisely the tangent line of the curve C_μ given by (32). Similarly, the tangent plane intersects the quadric Q_v in the v-tangent, $x_4 = x_2 = 0$, and in the same residual line (32). Therefore, *the conjugate chord quadric Q_u (Q_v) at every point of a curve C on a conjugate net intersects the tangent plane of the net in the u-tangent (v-tangent) and in a residual line tangent to the curve in the direction conjugate to the curve C.*

It is well known that a necessary and sufficient condition for a direction to coincide with its conjugate is that it be an asymptotic direction. Hence, *the residual line in which the tangent plane of the net intersects the two conjugate chord quadrics at every point of a curve on a net coincides with the tangent to the curve if, and only if, the curve is an asymptotic curve on the sustaining surface.*

Lane has shown [5] that the conjugate osculating quadric Q_u is intersected by the tangent plane in the u-tangent and in the residual line whose equation, referred to the same tetrahedron that we are using, is

$$x_4 = (L - N\lambda^2)x_3 - 2\lambda L x_2 = 0. \tag{33}$$

Similarly, the tangent plane intersects the conjugate osculating quadric Q_v in the v-tangent and in a residual line in general distinct from the line (33). The two residual lines in which the tangent plane intersects the two conjugate osculating quadrics coincide [6] if, and only if, the curve C_λ is an asymptotic curve on S, and in this case the lines coincide with the tangent of the curve.

The residual line (32) of the conjugate chord quadric Q_u coincides with the residual line (33) of the conjugate osculating quadric Q_u if, and only if, $1 + r\lambda^2 = 0$, i. e., if C_λ is an asymptotic curve on S. Therefore, *the conjugate chord quadrics and the conjugate osculating quadrics at every point of a curve on a net are intersected by the tangent plane of the net in the same lines if, and only if, the curve is an asymptotic curve on the surface; then the lines are the u-tangent, the v-tangent, and the tangent of the curve.*

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[5] Lane, "Conjugate nets," p. 579.

[6] *Ibid.*, p. 580.

We next consider briefly the intersections of the conjugate chord quadrics with the two osculating planes $x_3 = 0$, $x_2 = 0$, of the curves C_u, C_v respectively. One of the generators of the quadric Q_u, in the osculating plane $x_3 = 0$ at the point P_x of the curve C_u, is the u-tangent, $x_3 = x_4 = 0$. Another generator in this plane is the line

$$(\mathfrak{D} + 2\lambda K - 2\boldsymbol{H}/3\lambda r)x_4 - 2Lx_1 = 0. \tag{34}$$

In the osculating plane $x_3 = 0$, one of the generators of the conjugate osculating quadric Q_u is found [7] to be the u-tangent also, and the line

$$(\mathfrak{D} + \lambda K - \boldsymbol{H}/\lambda r)x_4 - 2Lx_1 = 0. \tag{35}$$

Thus, on comparing equations (34) and (35) we find that *the osculating plane $x_3 = 0$, at a point P_x of a curve C_u, contains the tangent of this curve as a common generator of the conjugate chord quadric Q_u and the conjugate osculating quadric Q_u at a point P_x of a curve C_λ. The other generators, in the plane $x_3 = 0$, of these two quadrics coincide if, and only if,*

$$\lambda^2 = -\boldsymbol{H}/3rK.$$

As is also true in the case of the conjugate osculating quadric Q_u, the osculating plane, $x_3 = 0$, touches the conjugate chord quadric Q_u of the curve C_λ in the ray-point ρ of the curve C_v.

Results similar to the preceding may be obtained by considering the quadric Q_v and the osculating plane, $x_2 = 0$, of the curve C_v.

The equation of the cone projecting from the point P_x the curve of intersection of the two conjugate chord quadrics Q_u, Q_v is found by eliminating x_1 from their equations. For this result we find

$$\begin{aligned} &L(x_3 - \lambda x_2)(x_2 + r\lambda x_3) + [4(\mathfrak{C}'/3 + \lambda\mathfrak{B}') - \lambda(\log \lambda r^{1/2})']x_3x_4 \\ &\quad - \lambda[4(\mathfrak{C}' + \lambda\mathfrak{B}'/3) + (\log \lambda r^{1/2})']x_2x_4 \\ &\quad + 2[\lambda\mathfrak{D}/L - (H + \boldsymbol{H}/3)/N + \lambda^2(K + \boldsymbol{K}/3)/L]x_4^2 = 0. \end{aligned} \tag{36}$$

This cone will be tangent to the tangent plane, $x_4 = 0$, along the tangent line (31) of the curve C_λ in case $1 + r\lambda^2 = 0$, that is, in case the curve C_λ is an asymptotic curve.

It is of some interest to consider the relations of the two conjugate chord quadrics Q_u for the curves C_λ, C_μ of the two conjugate families. The equation of the quadric Q_u for the curve C_μ, with μ determined by equation (30), can be written immediately by replacing λ by $-1/r\lambda$ in equation (27). This result is found to be

[7] *Ibid.*, p. 580.

$$(37)\qquad \begin{aligned} &Nx_3^2 + [4(\mathfrak{B}' - r\lambda\mathfrak{C}'/3) + (\log \lambda r^{1/2})_u - (\log \lambda r^{1/2})_v/r\lambda]x_3x_4 \\ &\quad - 2[n + K/\lambda N - \lambda \boldsymbol{H}/3L]x_4^2 - \lambda N x_2x_3 - 2x_1x_4 = 0. \end{aligned}$$

If x_1 is eliminated from equations (27) and (37), we obtain the equation of the cone projecting from the point P_x the curve of intersection of the conjugate chord quadrics Q_u for the curves of the two conjugate families, namely,

$$(38)\qquad \begin{aligned} &[4(1 + r\lambda^2)r\mathfrak{C}'/3 - 2r\lambda(\log \lambda r^{1/2})_u + (1 - r\lambda^2)(\log \lambda r^{1/2})_v]x_3x_4 \\ &\quad + N(1 + r\lambda^2)x_2x_3 + 2(1 + r\lambda^2)(K - \boldsymbol{H}/3)x_4^2/L = 0. \end{aligned}$$

This cone is indeterminate in case $1 + r\lambda^2 = 0$, that is, when the curve C_λ is an asymptotic curve. In this case the two quadrics (27) and (37) coincide. In case $1 + r\lambda^2 \neq 0$ and $\boldsymbol{H} = 3K$ the cone (38) is a pair of planes so that the two quadrics intersect in two conics. One of the planes is the osculating plane, $x_3 = 0$, of the curve C_u, and the equation of the other is easily read from equation (38). Otherwise the cone (38) is tangent to the osculating plane, $x_3 = 0$, of the curve C_u, touching it along its tangent line, $x_3 = x_4 = 0$.

SOUTHWESTERN COLLEGE,
MEMPHIS, TENN.

ARITHMETICAL THEOREMS ON LUCAS FUNCTIONS AND TCHEBYCHEFF POLYNOMIALS.

By E. T. Bell.

1. *Introduction and Summary.* Any identity in elliptic, or elliptic theta functions, is equivalent to (implies and is implied by) an identity in Lucas [1] functions or Tchebycheff polynomials with arbitrary complex arguments. We first develop an isomorphism between the Lucas functions and circular functions.

2. *The functions U, V, C, S, T.* Principal values of all irrationalities, wherever they occur, are to be understood. Let α, β, z, u be complex variables such that $\alpha\beta(\alpha-\beta)\neq 0$. Write $b\equiv\alpha+\beta$, $c\equiv\alpha\beta$. Then α,β are the roots of

$$y^2-by+c=0; \tag{1}$$

$$2\alpha=b+(b^2-4c)^{1/2}, \qquad 2\beta=b-(b^2-4c)^{1/2}.$$

The functions U, V, associated with (1), are defined by

$$U(b,c,z)\equiv\frac{\alpha^z-\beta^z}{\alpha-\beta}, \qquad V(b,c,z)\equiv\alpha^z+\beta^z. \tag{2}$$

The connection with circular functions mentioned in § 1 will be made by means of

$$C(u,z)\equiv\tfrac{1}{2}V(2u,1,z), \qquad S(u,z)\equiv U(2u,1,z). \tag{3}$$

Thus $C(u,z)$, $S(u,z)$ are associated with

$$y^2-2uy+1=0, \tag{4}$$

of which the roots are

$$\rho=u+(u^2-1)^{1/2}, \qquad \sigma=u-(u^2-1)^{1/2},$$

and we have

$$C(u,z)=\tfrac{1}{2}(\rho^z+\sigma^z), \qquad S(u,z)=\frac{\rho^z-\sigma^z}{\rho-\sigma}.$$

Comparing the last with (1), (2), we have the important formulas

[1] The functions are the U_n, V_n of Lucas, *American Journal of Mathematics*, vol. 1 (1878), pp. 184-240, 288-321; the arguments are his P, Q, which he restricts to be rational. The designation "Tchebycheff Polynomials" is retained as it appears to be accepted, although these polynomials occur in the writings of Vieta, Newton, De Moivre, Johann Bernoulli and Euler, to mention only predecessors of Gauss and Cauchy. According to Lucas, *loc. cit.*, p. 208, they first appear in Vieta's *Opera*, Leyden, 1646, pp. 295-299.

$$(5)\qquad U(b,c,z)=c^{(z-1)/2}\,S(b/2c^{1/2},z),\qquad V(b,c,z)=2c^{z/2}\,C(b/2c^{1/2},z).$$

To connect our functions with one usual definition (for example that of Polya and Szegö) of the Tchebycheff polynomials we introduce

$$(6)\qquad T(u,z)\equiv S(u,z+1),$$

which, for z an integer, is a Tchebycheff polynomial. Only one of S, T is necessary, as any formula involving either may be restated immediately in terms of the other by means of (6). We shall use S in preference to T.

From (2) we have

$$(7)\quad U(b,c,0)=0,\quad U(b,c,1)=1;\qquad V(b,c,0)=2,\quad V(b,c,1)=b;$$

$$(8)\quad U(b,c,-z)=-c^{-z}\,U(b,c,z);\qquad V(b,c,-z)=c^{-z}\,V(b,c,z);$$

$$(9)\quad W(b,c,z+n+2)-bW(b,c,z+n+1)+cW(b,c,z+n)=0,$$
$$W=U,V;\ n=0,\pm1,\pm2,\cdots.$$

From (7), (9) the $U(b,c,z+n)$, $V(b,c,z+n)$ can be calculated by recurrence. From these and (3), (4) we write down the corresponding relations for C, S:

$$(10)\quad C(u,0)=1,\quad C(u,1)=u;\qquad S(u,0)=0,\quad S(u,1)=1;$$

$$(11)\quad C(u,-z)=C(u,z);\qquad S(u,-z)=-S(u,z);$$

$$(12)\quad Z(u,z+n+2)-2uZ(u,z+n+1)+Z(u,z+n)=0,$$
$$Z=C,S;\quad n=0,\pm1,\pm2,\cdots.$$

Let m be an integer >0. Then, identically,

$$\cos(m+2)\theta-2\cos\theta\cos(m+1)\theta+\cos m\theta=0,$$
$$\sin(m+2)\theta-2\cos\theta\sin(m+1)\theta+\sin m\theta=0.$$

Comparing the first of these with (10), (12), we see that $C(u,m)$ is the Tchebycheff polynomial obtained from the expansion of $\cos m\theta$ in powers of $\cos\theta$ on replacing $\cos\theta$ by u. From the second, $S(u,m+1)$ is the Tchebycheff polynomial $T(u,m)$ obtained on replacing $\cos\theta$ by u in the expansion of $\sin(m+1)\theta\csc\theta$ in powers of $\cos\theta$. Hence, for integer values of $m>0$, we have the explicit forms

$$(13)\qquad C(u,m)=m\sum_{r=0}^{[(m-1)/2]}\frac{(-1)^r}{m-2r}\binom{m-r-1}{r}2^{m-2r-1}u^{m-2r},$$

$$(14)\qquad S(u,m)=u\sum_{r=0}^{[(m-1)/2]}(-1)^r\binom{m-r-1}{r}2^{m-2r-1}u^{m-2r},$$

where $[x]$ is the greatest integer in x. These are in descending powers of u. In ascending powers we have

$$(15)\quad C(u,2m) = (-1)^m\, m \sum_{s=0}^{m} \frac{(-1)^s 2^{2s}}{m+s} \binom{m+s}{2s} u^{2s},$$

$$(16)\quad C(u,2m-1) = (-1)^{m-1}\,(2m-1)u \sum_{s=0}^{m-1} \frac{(-1)^s 2^{2s}}{m+s} \binom{m+s}{2s+1} u^{2s},$$

$$(17)\quad S(u,2m) = 2(-1)^{m+1} u \sum_{s=0}^{m-1} (-1)^s 2^{2s} \binom{m+s}{2s+1} u^{2s},$$

$$(18)\quad S(u,2m-1) = (-1)^{m-1} \sum_{s=0}^{m-1} (-1)^s 2^{2s} \binom{m+s-1}{2s} u^{2s}.$$

To write down the expressions of $S(u,n)$, $C(u,n)$, where n is an arbitrary integer (positive, zero, or negative) we refer to (10), (11) and use the sgn function, defined for all real values of x by $\operatorname{sgn} x = -1, 0, 1$ according as $x<0$, $x=0$, $x>0$. Then

$$(19)\qquad C(u,n) = C(u,|n|), \qquad S(u,n) = \operatorname{sgn} n\, S(u,|n|).$$

From (13)-(18) and (5) we have the following reduced forms of the $U(b,c,m)$, $V(b,c,m)$ for all integers $m>0$:

$$(20)\quad V(b,c,m) = \sum_{r=0}^{[(m-1)/2]} \frac{(-1)^r m}{m-2r} \binom{m-r-1}{r} b^{m-2r} c^r,$$

$$(21)\quad U(b,c,m) = b \sum_{r=0}^{[(m-1)/2]} (-1)^r \binom{m-r-1}{r} b^{m-2r} c^r;$$

$$(22)\quad V(b,c,2m) = 2(-1)^m\, m \sum_{s=0}^{m} \frac{(-1)^s}{m+s} \binom{m+s}{2s} b^{2s} c^{m-s},$$

$$(23)\quad V(b,c,2m-1) = (-1)^{m-1}(2m-1) b \sum_{s=0}^{m-1} \frac{(-1)^s}{m+s} \binom{m+s}{2s+1} b^{2s} c^{m-s-1},$$

$$(24)\quad U(b,c,2m) = (-1)^{m+1} b \sum_{s=0}^{m-1} (-1)^s \binom{m+s}{2s+1} b^{2s} c^{m-s-1},$$

$$(25)\quad U(b,c,2m-1) = (-1)^{m-1} \sum_{s=0}^{m-1} (-1)^s \binom{m+s-1}{2s} b^{2s} c^{m-s-1}.$$

Corresponding to (19), for arbitrary integers n,

$$(26)\qquad \begin{aligned} V(b,c,n) &= c^{n(\operatorname{sgn} n - 1)/2}\, V(b,c,|n|), \\ U(b,c,n) &= \operatorname{sgn} n\, c^{n(\operatorname{sgn} n - 1)/2}\, U(b,c,|n|), \end{aligned}$$

which refer all cases to (20)-(26). As (22)-(25) are merely (20), (21) written in reverse order, the numerical coefficients are integers.

We shall call $U(b,c,n)$, $V(b,c,n)$ the Lucas functions of the arguments b, c, of order n. Lucas had occasion to consider his functions only for rational (for the most part integral) arguments. The generating functions of $U(b,c,n)$,

$V(b, c, n)$ can be written down from (2), if needed; the information given by them is, however, more readily obtainable from the isomorphism next noted.

3. *Trigonometric identities.* Lucas (*loc. cit.*, p. 189) emphasized the connection between his functions and the circular (or hyperbolic) functions, and he gave a method for passing from identities in his functions to identities in circular functions. The method is also reversible. We shall give another way of accomplishing these things, which seems more direct and which is much easier to apply. What follows is generalized in § 5.

Let the n_i, r_j be integers, and let $P(z_1, \cdots, z_{s+t})$ be a polynomial in $z_1, \cdots, z_{s+t}$. For $0 < x < \pi/2$ let

$$P(\cos n_1 x, \cdots, \cos n_s x, \sin r_1 x, \cdots, \sin r_t x) = 0$$

be an identity in x. Replace $\sin rx$ by $\sin x \sin r_j x \csc x$ $(j = 1, \cdots, t)$. The result is an identity in x for the same range. In this we write

$$\cos n_i x = C(u, n_i), \qquad \sin r_j x \csc x = S(u, r_j),$$
$$u = \cos x, \qquad \sin x = (1 - u^2)^{1/2} \qquad (i = 1, \cdots, s;\ j = 1, \cdots, t);$$

say the result is $R(u) = 0$. Then $R(u) = 0$ is an identity in u, since $R(u)$ is a polynomial in u, $(1 - u^2)^{1/2}$ which vanishes for an infinity of distinct values of u. Hence in $R(u) = 0$ we may replace u by the complex variable w, getting the identity $R(w) = 0$ in w. Choose $w = b/2c^{1/2}$ (as in (5)). It follows that we can pass directly from the original cos, sin identity $P(\ ) = 0$ to $P(\cdots, 2c^{-n_i/2} V(b, c, n_i), \cdots, \cdots, \frac{1}{2}(4c - b^2)^{1/2} c^{-r_j/2} U(b, c, r_j), \cdots) = 0$. This converts the trigonometric identity $P(\ ) = 0$ into an identity in functions U, V with the arbitrary arguments b, c. The process is evidently reversible, and we may sum up the simple algorithm in the following table, in which n is an integer. (See also § 5.)

Circular	*Lucas*
$\cos nx$	$\frac{1}{2} c^{-n/2} V(b, c, n)$
$\sin nx \csc x$	$c^{(1-n)/2} U(b, c, n)$
$\sin x$	$(1 - b^2/4c)^{1/2}$
$\cos x$	$b/2c^{1/2}$

To pass from a trigonometric identity in x to its correspondent in Lucas functions we replace the functions in the first column by their correspondents in the second. To pass from an identity in Lucas functions to its correspondent in circular functions, we replace $V(b, c, n)$ by $2c^{n/2} \cos nx$, and $U(b, c, n)$ by $c^{(n-1)/2} \sin nx \csc x$. The last two lines of the table suffice to eliminate b, c

from the result, if they are not already absent; if not, the supposed identity is untrue.

We give some simple examples. All admit of generalization. From (m, n are integers)

$$\sin (m+n)x = \sin mx \cos nx + \cos mx \sin nx,$$

and the table, we have

$$2U(b,c,m+n) = U(b,c,m)V(b,c,n) + V(b,c,m)U(b,c,n).$$

Similarly

$$\cos (m+n)x = \cos mx \cos nx - \sin^2 x \frac{\sin mx}{\sin x} \frac{\sin nx}{\sin x},$$

$$2V(b,c,m+n) = V(b,c,m)V(b,c,n) + (b^2-4c)U(b,c,m)U(b,c,n).$$

To indicate the inverse process we verify the evident identity

$$U(b,c,2n) = U(b,c,n)V(b,c,n),$$

which is obtained from the first of the above by taking $m = n$. If the identity is true, then

$$c^{(2n-1)/2} \sin 2nx \csc x = c^{(2n-1)/2} \sin nx \csc x \, 2c^{n/2} \cos nx,$$

which is $\sin 2nx = 2 \sin nx \cos nx$, and hence the identity is verified, since the steps are evidently reversible.

The algorithm provides a straightforward way for deriving relations between the U, V. Suppose, for example, we wish to sum

$$\sum_{s=1}^{n} V^p(b,c,s)$$

in terms of functions U, V, where p is a positive integer. By the table, this is equivalent to summing

$$2^p \sum_{s=1}^{n} (c^{p/2})^s \cos^p sx,$$

expressing the result as a function of sines and cosines, and reapplying the table to replace sin, cos by U, V. The special cases of this considered by Lucas follow in this way at once.

The isomorphism expressed in the table also suggests a possible generalization of the U, V which it might be profitable to investigate: the two functions U, V are replaced by the four polynomials in $\operatorname{sn} x$, $\operatorname{cn} x$ occurring in the numerators and denominators of $\operatorname{sn} nx$, $\operatorname{cn} nx$, $\operatorname{dn} nx$. When the modulus is zero, the polynomials degenerate to the U, V.

The isomorphism also leads to the theory of "division" for the U, V, which Lucas mentions in passing, but on which he did not, apparently, publish anything. This concerns the expression of $U(b, c, t)$, $V(b, c, t)$ with t rational but not integral, as algebraic functions (in the usual technical sense as in complex variables) of U's and V's of integral order. As this does not concern the application to elliptic functions which we have in view here, we shall not go into it.

If the identities involve several sets of variables x_i, b_i, c_i and associated integers n_{ij}, the table is applied to each. There is an example in § 4.

4. *Elliptic and theta identities.* It was shown in a previous paper [2] that any identity in elliptic or elliptic theta functions is equivalent to an identity in arbitrary parity functions summed over one or more quadratic partitions of integers. The parity functions may be replaced by products of circular functions (sines or cosines) having the same parity. This identity is also equivalent to the original identity. Transforming the circular identity by means of the table in § 3 into its equivalent in U, V functions, we get an identity in Lucas functions which is equivalent to the elliptic identity. We need give only enough examples [3] to illustrate the process.

From the expansion of $\vartheta_1'/\vartheta_0^2(x)$ in arithmetical form, as given in M § 10, we have

$$\vartheta_1'^2/\vartheta_0^2(x) = 4\Sigma q^{m/2}[\Sigma t \cos{(t-\tau)x}] \qquad (m = 1, 3, 5, \cdots;\ m = t\tau,\ t > 0,\ \tau > 0).$$

Hence, using

$$\vartheta_0(x) = \Sigma q^{\nu^2}(-1)^\nu \cos 2\nu x \cdot \qquad (\nu = 0, \pm 1, \pm 2, \cdots),$$

multiplying throughout by $\vartheta_0^2(x)$, and applying

$$\vartheta_1'^2 = [\Sigma(-1)^{(\mu-1)/2}\mu q^{\mu^2/4}]^2, \qquad (\mu = \pm 1, \pm 3, \pm 5, \cdots),$$

we have the following identity on equating coefficients of like powers of q:

$$\Sigma t \cos(t - \tau + 2\nu_1 + 2\nu_2)x = \Sigma(-1)^{(m_1 m_2 - 1)/2} m_1 m_2,$$

where the sums refer to all positive integers t, τ, m_1, m_2, and to all integers ν_1, ν_2 (positive, zero, or negative) such that, for n a constant integer $\geqq 0$,

$$4n + 2 = m_1^2 + m_2^2 = t\tau + 2\nu_1^2 + 2\nu_2^2.$$

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[2] *Transactions of the American Mathematical Society*, vol. 22 (1921), p. 1; cited as A.

[3] The expansions used are quoted either from the paper cited in the preceding footnote, or from my paper in *Messenger of Mathematics*, vol. 54 (1924), pp. 166-176, cited as M.

Applying the table in § 3, we have

$$\Sigma t c^{(\tau-t-2\nu_1-2\nu_2)} V(b, c, t-\tau+2\nu_1+2\nu_2) = 2\Sigma(-1)^{(m_1m_2-1)/2} m_1 m_2,$$

where the sums are as just defined and b, c are independent complex variables. The other identity of this sort is obtained from $\vartheta_1'^2/\vartheta_2^2(x)$, by applying the method of A to the expansion in M § 10.

As a somewhat simpler specimen of the infinity of results obtainable from the available expansions we shall state the equivalent of the identity $[\vartheta_1'/\vartheta_3(x)]\vartheta_3(x) = \vartheta_1'$, using the expansion of $\vartheta_1'/\vartheta_1(x)$ in M § 8.

Let $(-1 \mid m)$, $(2 \mid m)$ be defined as Jacobi-Legendre symbols when m is an odd integer, and write $\epsilon(n) = 1$, or 0, according as n is, or is not, the square of an integer > 0. Consider all the integer solutions, for α constant and $\equiv 1 \bmod 4$, of

$$\alpha = t\tau + 4\nu^2 \qquad (t > 0,\ \tau > 0,\ \nu \gtreqless 0),$$

and let Σ refer to all sets of solutions (t, τ, ν). Then

$$4\Sigma(-1 \mid \tau)(2 \mid t\tau) c^{(\tau-t-4\nu)/4} V(b, c, t-\tau+4\nu) = \epsilon(m)(-1 \mid m^{1/2}) m^{1/2}.$$

Such identities can be given a form analogous to that of the recurrences for functions of divisors, of which the first example is due to Euler. Thus, from M § 8, we have

$$\vartheta_1'/\vartheta_0(x) = 2\Sigma q^{\alpha/4}\left[\Sigma(-1 \mid \tau)\cos\left(\frac{t-\tau}{2}\right)x\right],$$

where the first Σ refers to $\alpha = 1, 5, 9, 13, \cdots$, and the second to all positive integers t, τ such that $\alpha = t\tau$. Write

$$P(b, c, \alpha) \equiv \Sigma(-1 \mid \tau) c^{(\tau-t)/2} V\left(b, c, \frac{t-\tau}{2}\right),$$

where Σ refers to t, τ, and $t\tau = \alpha$ (as before). Then the identity $[\vartheta_1'/\vartheta_0(x)]\vartheta_0(x) = \vartheta_1'$ is equivalent to

$$2\Sigma(-c)^{-\nu} P(b, c, m-4\nu^2) = \epsilon(m)(-1 \mid m^{1/2}) m^{1/2},$$

where m is positive and $\equiv 1 \bmod 4$, and Σ refers to $\nu = 0, \pm 1, \pm 2, \cdots$, the sum continuing so long as $4\nu^2 < m$.

To illustrate identities in more than one variable we take $f(x, y) \equiv \cos \xi x \cos x\eta y$ in a theorem in A,

$$2\Sigma[f(d_1-d_2, \delta_1+\delta_2) - f(d_1+d_2, \delta_1-\delta_2)] = \Sigma t[f(0, 2t) - f(2t, 0)]$$

in which $f(x, y)$ is an arbitrary single-valued function of x, y satisfying the conditions $f(x, y) = f(-x, y) = f(x, -y)$, and the sums refer to all positive

integers d_1, δ_1, d_2, δ_2, t, τ such that $2t\tau = d_1\delta_1 + d_2\delta_2$, τ, d_1, δ_1, d_2, δ_2 are odd, and $t\tau$ is constant. The equivalent in Lucas functions is

$$\begin{aligned}\Sigma[&y_1^{(d_2-d_1)/2}y_2^{-(\delta_1+\delta_2)/2}V(x_1, y_1, d_2-\delta_2)\,V(x_2, y_2, \delta_1+\delta_2)\\ &-y_1^{-(d_1+d_2)/2}y_2^{-(\delta_1-\delta_2)/2}V(x_1, y_1, d_2+d_2)\,V(x_2, y_2, \delta_1-\delta_2)]\\ &=2\Sigma[y_2^{-t}V(x_2, y_2, 2t)-y_1^{-t}V(x_1, y_1, 2t)],\end{aligned}$$

5. *Generalized isomorphism.* Let x be a real or complex variable, and z a complex variable. Then the isomorphism may be generalized as follows, in which $\sim$ is the sign of 1, 1 correspondence:

$$\cos x \sim \tfrac{1}{2}c^{-z/2}V(z), \qquad \sin x \sim \tfrac{1}{2}c^{-z/2}\Delta^{1/2}U(z),$$
$$V(z) \equiv V(b, c, z), \quad U(z) \equiv U(b, c, z), \quad \Delta \equiv \Delta(b, c) \equiv 4c - b^2,$$

in which b, c as before are complex variables. For, the variables α, β in a U, V relation may be replaced by other variables, or by functions of themselves. Under the particular transformation $\alpha \to \alpha^z$, $\beta \to \beta^z$, we have, from the original isomorphism,

$$\cos nx \sim \tfrac{1}{2}c^{-nz/2}V(nz), \qquad \sin nx \sim \tfrac{1}{2}c^{-nz/2}\Delta^{1/2}(nz),$$

as may be readily verified, in which n is an arbitrary integer. Taking $n = 1$ we get the form stated.

Let $x_1, \cdots, x_s$ be independent real or complex variables, and $z_1, \cdots, z_s$ independent complex variables. Then, more generally, we have the above isomorphism in which now

$$x \equiv x_1 + \cdots + x_s, \qquad z \equiv z_1 + \cdots + z_s,$$

and the correspondence between the two sets of independent variables is $x_j \sim z_j$ $(j = 1, \cdots, s)$. This is evident upon mathematical induction on s applied to the addition theorems for cos, sin, V, U, and an application of the isomorphism as stated at the beginning of this section. Finally, in the same way, or as a consequence, we have the isomorphism in which

$$x \equiv n_1x_1 + \cdots + n_sx_s, \qquad z \equiv n_1z_1 + \cdots + n_sz_s,$$

where the n's are integers, and $n_jx_j \sim n_jz_j$ $(j = 1, \cdots, s)$.

California Institute of Technology.

BINARY QUADRATIC DISCRIMINANTS DIFFERING BY SQUARE FACTORS.

By GORDON PALL.

1. The problem of finding the number of representations of numbers not prime to the discriminant in an integral quadratic form, or in a genus or order thereof, has never been adequately solved. The classical methods depending on forms constructed from the roots of certain congruences, subject to conditions which isolate the forms of a given class, genus, or order, are simple enough for numbers prime to the discriminant, but are excessively detailed and tedious in general. One reason for these complications is that those methods lead first to proper representations, and through them to all, and the relations between proper representations in different classes and genera are more involved than those for all representations. To avoid the difficulty we may attempt to reduce the problem to one in which the order-invariants are (so far as possible) free of square factors.

We treat in this paper binary quadratic forms, our principal method employing reduction formulae to replace a problem concerning forms of discriminant p^2d by a like problem for discriminant d.

A discriminant d is a non-zero integer congruent to 0 or 1 (mod 4), and is fundamental if divisible by no prime p such that d/p^2 is a discriminant, i. e.

$$p > 2 \text{ and } p^2 \mid d, \text{ or } p = 2 \text{ and } d \equiv 0 \text{ or } 4 \pmod{16}. \tag{1}$$

Let $r(n, d)$ denote the number of sets [1] of representations of n in a representative system of primitive integral binary quadratic forms of discriminant d. Then $r(n, d)$ is a factorable function of n, and if p does not satisfy (1),

$$r(p^a, d) = 1 + (d \mid p) + (d \mid p^2) + \cdots + (d \mid p^a). \tag{2}$$ [2]

It is also known that

$$r(p, d) = 0 \text{ if } p \text{ satisfies } (1). \tag{3}$$ [2]

Consequently a prime is represented in at most one class (and the reciprocal class) of discriminant d. On the basis of this property and the composition of classes the writer has elsewhere (Pall I and II) considered the

[1] Pall II, 491; or Dirichlet, 216.

[2] Pall II, p. 493.

number $f(n)$ of sets of representations of n in any integral binary quadratic form f, but only partially for n divisible by primes of type (1).

To any primitive f_1 of discriminant p^2d corresponds a form f of discriminant d such that

$$f_1(p^2n) = \sigma f(n), \text{ for every integer } n.^3 \tag{4}$$

Here $\sigma = \sigma_{p,d}$ is defined as follows:

(5) $\sigma = 1$ if $d < -4$ or d is square, $\sigma = 2$ if $d = -4$, $\sigma = 3$ if $d = -3$;

and if d is positive but not square, σ is the least positive integer such that $p \mid u_\sigma$, where (t_k, u_k) denote the successive integral solutions of

$$t^2 - du^2 = 4. \tag{6}$$

In § 5 we shall associate (4) with the Lipschitz correspondence between the classes of discriminants d and p^2d.[4] The formula

$$h(p^2d)/h(d) = \{p - (d \mid p)\}/\sigma, \tag{7}$$

which was the object of Lipschitz's investigation, can be deduced directly from (4) and (9), $h(d)$ denoting the number of primitive classes of discriminant d.[5]

The slight variant of Lipschitz's method, given in § 2, may be of value. Lemma 3 replaces the reference to Gauss' complicated Art. 162, and finds further applications in § 3.

A primitive ambiguous class of discriminant d has ν derived primitive ambiguous classes of discriminant p^2d, where ν is given in Theorem 4. We note in § 3 the following curious corollary:

Frequently, an ambiguous class C of even discriminant d contains one ambiguous form $[a, b, c]$ with b/a even, and another with b/a odd. If $\sigma = \sigma_{2,d} > 1$ and $d \equiv 0$, 12, or 28 (mod 32) such forms of both kinds seem to exist in every primitive ambiguous class; if $d \equiv 4$ or 8 (mod 16), or 16 (mod 32), then $\sigma = 1$, and it is trivial that every primitive ambiguous form has b/a even. But:

(8) if $d \equiv 0$, 12, or 28 (mod 32) and $\sigma = 1$, every primitive ambiguous class contains ambiguous forms of only one kind, and there are equally many ambiguous classes of either kind.

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[3] Pall I, p. 331, II, p. 494.
[4] Lipschitz, p. 255.
[5] Pall III.

Further properties of derived classes appear in section 4.

Finally we note in section 6 that

$$r(p^2n, p^2d) = \{p - (d \mid p)\}r(n, d), \tag{9}$$

which together with (2) and (3) yields immediately the value of $r(p^a, d)$ for any p.

If n has no factor of type (1), n is represented in at most one primitive genus of discriminant d. In any case it is represented equally often in each of the 2^t genera which represent it, specified in § 7.

2. Binary matrices P and Q are called *right-equivalent* if $PU = Q$ for some unitary integral matrix U; *left-equivalent* if $UP = Q$.

(10) If $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $P^\dagger$ denotes the adjoint-transpose $\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$.

Evidently $PU = Q$ is equivalent to $U^\dagger P^\dagger = Q^\dagger$, and every P is the adjoint-transpose of the matrix $P^\dagger$ of the same determinant. We have

LEMMA 1. *Every integral matrix of prime determinant p is right-equivalent to one and only one of the $p + 1$ matrices*

$$P_\kappa = \begin{pmatrix} p & \kappa \\ 0 & 1 \end{pmatrix} \quad (\kappa = 0, 1, \cdots, p-1), \qquad P_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}; \tag{11}$$

[6]

and left-equivalent to one and only one of

$$P_\kappa^\dagger = \begin{pmatrix} 1 & -\kappa \\ 0 & p \end{pmatrix} \quad (\kappa = 0, 1, \cdots, p-1), \qquad P_p^\dagger = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}. \tag{12}$$

By fP we denote the form derived from $f = [a, b, c]$ by applying the linear transformation of matrix P. By the associativity of matrix multiplication, $f(PU) = fPU$. Hence

THEOREM 1. *Let $f = [a, b, c]$ be primitive, $b^2 - 4ac = d \neq 0$. Every form derivable from f by integral transformations of prime determinant p is equivalent to one of the $p + 1$ forms*

$$\begin{aligned} fP_\kappa &= [ap^2, (2a\kappa + b)p, a\kappa^2 + b\kappa + c], \qquad (\kappa = 0, \cdots, p-1), \\ fP_p &= [a, bp, cp^2]. \end{aligned} \tag{13}$$

Exactly $p - (d \mid p)$ of the forms (13) are primitive.

[6] Mathews, p. 160; or Smith, pp. 166, 176.

The $1 + (d \mid p)$ imprimitive forms fP_κ are given by

$$a\kappa^2 + b\kappa + c \equiv 0 \pmod{p} \qquad (0 \leqq \kappa < p) \tag{14}$$

if $p \nmid a$; by $\kappa = p$ and (14) if $p \mid a$. In every case the divisor of the imprimitive fP_κ is p unless (1) holds, and then p^2.

Let V be an unitary integral matrix. If VP_κ is right-equivalent to VP_λ, $VP_\kappa = VP_\lambda U$, $P_\kappa = P_\lambda U$, $\kappa = \lambda$. Hence $VP_0, \cdots, VP_p$ are right-equivalent in some order to $P_0, \cdots, P_p$. This proves

LEMMA 2. *If f is replaced by an equivalent form g $(= fV)$, the classes of $gP_0, \cdots, gP_p$ constitute a permutation of the classes of $fP_0, \cdots, fP_p$.*

LEMMA 3. *Let f and g be integral primitive forms of discriminant d, P and Q integral binary matrices of determinant p. Let fP and gQ be equal and primitive. Then $R = (PQ^\dagger)/p$ is integral,* $\det R = 1$, $fR = g$.[7]

For choose integral unitary matrices U and V (Lemma 1) such that $U^\dagger P = P_\lambda{}^\dagger$ and $Q^\dagger V = P_\kappa$, and set $f' = fU$, $g' = gV$, $R' = (P_\lambda{}^\dagger P_\kappa)/p$. Then $R' = U^\dagger R V$, $f'R' = g'$, f' and g' are integral and of discriminant d, $fP = f'U^\dagger P = f'P_\lambda{}^\dagger$, $gQ = g'V^\dagger Q = g'P_\kappa{}^\dagger$. It remains to prove that if R' is not integral, then $f'P_\lambda{}^\dagger$ is imprimitive. Now according as (α), $\kappa, \lambda < p$, (β) $\kappa = p > \lambda$, (γ) $\kappa < p = \lambda$, (δ) $\kappa = \lambda = p$, R' is equal to

$$\begin{pmatrix} 1 & (\kappa - \lambda)/p \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1/p & -\lambda \\ 0 & p \end{pmatrix}, \quad \begin{pmatrix} p & \kappa \\ 0 & 1/p \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

Let $f' = [a, b, c]$. In cases (α) and (β),

$$f'P_\lambda{}^\dagger = [a, -2a\lambda + bp, a\lambda^2 - bp\lambda + cp^2]$$

and is imprimitive if $p \mid a$. If R' is not integral in case (α), $p \nmid \mu = \kappa - \lambda$, $f'R' = [a, \cdots, a\mu^2/p^2 + b\mu/p + c]$ is integral only if $p \mid a$. In case (β) the same holds for $f'R' = [a/p^2, \cdots, \cdots]$. In case (γ), $f'P_\lambda{}^\dagger = [ap^2, bp, c]$ is imprimitive if $p \mid c$, $f'R' = [ap^2, \cdots, a\kappa^2 + b\kappa/p + c/p^2]$ is integral only if $p \mid c$.

THEOREM 2. *Let d be a discriminant, p a prime.* (i) *Every primitive*

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[7] The conclusion may be false if $fP = gQ$ is imprimitive. If the divisor is p^2, (1) holds, and all we can say is that f and g are derived from the same class of discriminant d/p^2. If the divisor is p, p is represented in a class C, and C^{-1}, of discriminant d; then the classes of f and g are either identical, or one is C^2 times the other; this may be deduced from

$$fP_p/p = [a/p, b, cp] \sim [cp, -b, a/p] = [c, -b, a][p, -b, ac/p], \text{ etc.}$$

class K of discriminant p^2d is derivable from a primitive class C of discriminant d by an integral transformation of determinant p. (ii) *C is uniquely determined by K and p.*

(i) Replace d by p^2d in Theorem 1. The imprimitive form fP_κ is unique, and of divisor p^2; set $fP_\kappa = p^2f'$, whence f' is imprimitive and of discriminant d. Then

$$p^2f = f(P_\kappa P_\kappa^\dagger) = (fP_\kappa)P_\kappa^\dagger = p^2f'P_\kappa^\dagger, \qquad f = f'P_\kappa^\dagger. \tag{16}$$

(ii) The uniqueness follows from Lemma 3.

The proof of the fact that each of the $p - (d \mid p)$ *primitive* forms (13) is equivalent to exactly $\sigma - 1$ of the others, is based on the following criterion. If $fP \sim fQ$, $fPU = fQ$ for some unitary integral matrix U; by Lemma 3, $PUQ^\dagger/p$ is an integral unitary transformation A of f into f. Then $U = P^\dagger AQ/p$. Conversely, if $U \equiv P^\dagger AQ/p$ has all its elements integral, where A is any integral automorph of f, then $PU = AQ$, $fQ = fAQ = fPU \sim fP$.

The general integral automorph of $[a, b, c]$ (assumed primitive) is

$$A = \begin{pmatrix} \frac{1}{2}(t - bu) & -cu \\ au & \frac{1}{2}(t + bu) \end{pmatrix}, \tag{17}$$

where (t, u) runs over all integral solutions of (6). Forming $P_p^\dagger AP_\kappa$ $(\kappa < p)$ we find only one element not termwise divisible by p,

$$\Delta = au\kappa + \tfrac{1}{2}(t + bu). \tag{18}$$

To complete the proof of (7) it now suffices to prove

LEMMA 4. *If fP_p is primitive, i. e. $p \nmid a$, there are exactly $\sigma - 1$ values κ $(0 \leqq \kappa < p)$ such that $p \mid \Delta$ by choice of t, u.*

For by Lemma 2 it will follow for any primitive fP_κ, that there are precisely $\sigma - 1$ others equivalent to it in (13), by applying Lemma 4 to fV, where

$$V = \begin{pmatrix} p - \kappa & \kappa - p + 1 \\ -1 & 1 \end{pmatrix}, \text{ whence } fVP_p = fP_\kappa U \sim fP_\kappa, \; U \equiv \begin{pmatrix} 1 & 1 - p \\ -1 & p \end{pmatrix}.$$

If $u \equiv 0 \pmod p$, $t \equiv \pm 2$ if $p > 2$, and

$$\tfrac{1}{2}(t + bu) \equiv \tfrac{1}{4}(t^2 - b^2u^2) = 1 - acu^2 \equiv 1, \qquad \text{if } p = 2, \tag{19}$$

whence $\Delta \equiv \pm 1 \not\equiv 0$; disposing of all cases $\sigma = 1$ in Lemma 4.

If $d = -4$, we have also $t = 0$, $u = \pm 1$. If $f = [1, 0, 1]$, $\Delta = \pm\kappa$, $fP_p \sim fP_0$, and $\sigma = 2$ is effective in Lemma 4.

If $d=-3$, $f=[1,1,1]$, $t=\pm 1$, $u=\pm 1$, $\pm\Delta=\kappa+1$ or κ, $fP_p\sim fP_0\sim fP_{p-1}$, $\sigma=3$ is effective.

Finally consider d positive but not square, $\sigma>1$. Write

$$\Delta_k=au_k\kappa+\tfrac{1}{2}(t_k+bu_k), \tag{20}$$

whence

$$u_i\Delta_k-u_k\Delta_i=u_{i-k},\qquad t_i\Delta_k-t_k\Delta_i=(2a\kappa+b)u_{k-i}. \tag{21}$$

Now $p\mid u_i$ if and only if $\sigma\mid i$. If $i\equiv k\not\equiv 0\pmod\sigma$, $u_i\equiv\pm u_k\not\equiv 0\pmod p$, and by (21_1), $\Delta_i\equiv\pm\Delta_k\pmod p$. Thus we need consider only $k=1,\cdots,\sigma-1$. For each such k, $\Delta_k\equiv 0$ determines an unique κ $(0\leqq\kappa<p)$; and if $\Delta_i\equiv\Delta_k\equiv 0$ for the same κ, $p\mid u_{i-k}$ by (21_1), $\sigma\mid i-k$.

THEOREM 3. *Set $\tau=\{p-(d\mid p)\}/\sigma$. Exactly τ classes K correspond in Theorem 2 to each class C.*

3. With these developments freshly in mind we demonstrate

LEMMA 5. *Let $p>2$, $p\nmid a$, $d=b^2-4ac$ be positive but not square, $\sigma=\sigma_{p,d}>1$. Not counting the residue $\kappa\equiv-\frac{1}{2}b/a$ when $p\mid d$, the number of residues $\kappa\pmod p$ for which $\Delta_k\equiv\pm 1\pmod p$ by choice of $k=1,\cdots,\sigma-1$ is*

$$\sigma-1 \text{ if } p\mid d,\ 2(\sigma-1) \text{ if } p\nmid d.$$

i) Case $p>2$, $p\mid d$. By (6), $t_i\equiv\pm 2$ for every i. The residue $\kappa\equiv-\frac{1}{2}b/a$ appears from $\Delta_i\equiv 1$ if $t_i\equiv 2$, $\Delta_i\equiv-1$ if $t_i\equiv-2$. The remaining $\sigma-1$ residues κ determined by $\Delta_i\equiv\pm 1$ $(i=1,\cdots,\sigma-1)$ are distinct from each other and from $-\frac{1}{2}b/a$: for example, if $t_i\equiv 2\equiv-t_k$ and $\sigma\nmid i-k$, (21_2) shows that Δ_k and $-\Delta_i$ are incongruent for the same κ.

ii) Case $p>2$, $p\nmid d$. If $\Delta_i\equiv\pm 1$ and $\Delta_k\equiv 1$ or -1 for the same κ, then by (21_1), $u_{i-k}\equiv\pm u_i\pm u_k$, which is impossible unless $\sigma\mid i$, k, or $i-k$. [For example, if $i+k$ is even, $i=h+l$, $k=h-l$, $u_i+u_k=u_ht_l$, $u_{i-k}=u_it_l$, etc. If $i+k$ is odd, transpose a term, and proceed in the same way.]

THEOREM 4. *Let ν denote the number of primitive ambiguous classes of discriminant p^2d derivable from a primitive ambiguous class C of discriminant d, by integral transformations of determinant p. Then*

$$\nu=1 \text{ if } p>2,\ p\mid d;\ \nu=2 \text{ if } p>2,\ p\nmid d. \tag{22}$$

If $p=2$ let C contain the form $[a,b,c]$ in which $a\mid b$; then

$$\begin{aligned} &\nu = 0 \text{ if } d \equiv 0, 12, \text{ or } 28 \pmod{32}, \sigma = 1, \text{ and } b/a \text{ is odd};\\ (23)\quad &\nu = 2 \text{ if } b/a \text{ is even and } \sigma = 1;\\ &\nu = 1 \text{ if } d \text{ is odd, or } \sigma > 1. \end{aligned}$$

We may assume $a \mid b$ in f. For the theorem to hold, (13) should contain $\nu\sigma$ primitive fP_κ improperly equivalent to themselves. If $fP_\kappa = fP_\kappa V$, where V is an integral matrix of determinant -1, and B_1 is the matrix

$$\begin{pmatrix} 1 & b/a \\ 0 & -1 \end{pmatrix},$$

then $fB_1 = f$, $fP_\kappa = fB_1P_\kappa V$. By Lemma 3, if fP_κ is primitive, $B_1P_\kappa VP_\kappa{}'/p$ is an automorph (17); hence $P_\kappa VP_\kappa{}'/p$ is an improper integral automorph B of f, and $V = P_\kappa{}'BP_\kappa/p$. Conversely, if $V = P_\kappa{}'BP_\kappa/p$ is integral for some B, $fP_\kappa V = fBP_\kappa = fP_\kappa$, $fP_\kappa \sim fP_\kappa$.

The general improper integral automorph B of a primitive form $[a, b, c]$ in which $a \mid b$ is AB_1, with A as in (17). Replacing u by $-u$, we find

$$(24)\qquad B = \begin{pmatrix} \frac{1}{2}(t+bu) & \frac{1}{2}b(t+bu)/a - cu \\ -au & -\frac{1}{2}(t+bu) \end{pmatrix}.$$

The only element of $P_\kappa{}'BP_\kappa$ not obviously divisible by p is

$$(25)\qquad \Gamma = au\kappa^2 + (t+bu)\kappa + \tfrac{1}{2}(t+bu)(b/a) - cu.$$

I. Case $p > 2$, $p \mid a$. Then $p \mid b$, $p \nmid c$, and by (14) fP_κ is primitive ($\kappa < p$). Now $p \mid d$, $p \nmid t$ by (6), $p \mid \Gamma$ if and only if

$$\kappa + \tfrac{1}{2}b/a \equiv cu/t \pmod{p}.$$

As t, u range over solutions of (6), u/t has exactly σ residues, $\nu = 1$.

II. Case $p > 2$, $p \nmid a$. Then fP_p is primitive and ambiguous. The unique residue $\kappa \equiv -\frac{1}{2}b/a$ appears from $\Gamma \equiv 0$ when $t \equiv \pm 2$, $u \equiv 0$; then $a\kappa^2 + b\kappa + c \equiv -\frac{1}{4}d/a$, and by (14), fP_κ is primitive if $p \nmid d$, imprimitive if $p \mid d$. In the remaining cases, $p \nmid au$ and

$$(26)\qquad au\Gamma = (au\kappa + \tfrac{1}{2}(t+bu) + 1)(au\kappa + \tfrac{1}{2}(t+bu) - 1),$$

so that $\Gamma \equiv 0$ if and only if $\Delta \equiv \pm 1$. The residue $\kappa \equiv -\frac{1}{2}b/a$ cannot reappear, since it implies $4(\Delta^2 - 1) \equiv t^2 - 4 = du^2$. For d positive but not square (22) follows from Lemma 5. Our discussion is now complete also if $d < -4$ or d is a positive square. If $d = -4$, $f = [1, 0, 1]$, $\Delta = \kappa$, $fP_\kappa \sim fP_\kappa$ ($\kappa = 0, 1, p-1, p$), and since $\sigma = 2$, $\nu = 2$. If $d = -3$,

$f = [1, 1, 1]$, $\Delta = \kappa + 1$ or κ; if now $p > 3$, $fP_\kappa \sim fP_\kappa$ ($\kappa = 0$, $p-2$, 1, $p-1$, p, $\frac{1}{2}(p-1)$), and since $\sigma = 3$, $\nu = 2$; if $p = 3$, $\kappa = 0$, 2, 3 are effective, fP_1 being imprimitive, $\nu = 1$.[8]

III. Case $p = 2$, $\sigma = 1$ (every u even). Then $\frac{1}{2}(t + bu) \equiv 1$ by (19), $\Gamma \equiv b/a$. Hence $\nu = 0$ if b/a is odd and a even, $\nu = 1$ if b/a is odd and a odd (fP_2 primitive). If b/a is even, the two primitive forms (13) are non-equivalent ($\sigma = 1$), $\nu = 2$.

IV. Case $p = 2$, $\sigma > 1$. Since $du^2 + 4$ is a square with u odd,

$$d \equiv 5 \pmod 8,\ 12 \pmod{16},\ \text{or } 0 \pmod{32}. \tag{27}$$

In these cases, $\sigma = 2 - (d \mid 2) \geqq \nu\sigma$. Hence $\nu = 1$ if $\nu > 0$. If a is odd $\nu > 0$ since fP_2 is primitive; if a is even, d is even, the σ ($= 2$) primitive forms (13) are equivalent, $\Gamma \equiv u + b/a \pmod 2$ is even by choice of u, $\nu = 1$.

Remark. If $\gamma(d)$ denotes the number of primitive genera of discriminant d, and d has exactly t distinct prime factors,

$$\gamma(d) = \begin{cases} 2^{t-2} & \text{if } d \equiv 4 \pmod{16}, \\ 2^{t-1} & \text{if } d \equiv 1 \pmod 4,\ 8 \text{ or } 12 \pmod{16},\ 16 \pmod{32}, \\ 2^{t} & \text{if } d \equiv 0 \pmod{32}.[9] \end{cases} \tag{28}$$

Further the number of primitive ambiguous classes of discriminant d is equal to $\gamma(d)$. By (28),

$$\begin{aligned} \gamma(4d) &= \gamma(d) \text{ if } d \equiv 1 \pmod 4,\ 12 \pmod{16}, \text{ or } 0 \pmod{32}; \\ \gamma(4d) &= 2\gamma(d) \text{ if } d \equiv 4 \text{ or } 8 \pmod{16}, \text{ or } 16 \pmod{32}. \end{aligned} \tag{29}$$

Comparison of these facts with (23) yields (8).

For example, consider $d = -84$. Reduced forms:

$$f_0 = [1, 0, 21] \text{ and } f_1 = [3, 0, 7];\quad f_2 = [5, 4, 5] \text{ and } f_3 = [2, 2, 11].$$

From these are derived the following of discriminant $4d$:

$$\begin{gathered} f_0 \to [1, 0, 84], [4, 0, 21];\quad f_1 \to [3, 0, 28], [7, 0, 12]; \\ f_2 \to [5, \pm 2, 17];\quad f_3 \to [8, \pm 4, 11]. \end{gathered}$$

The least positive non-square $d \equiv 0$, 12, or 28 (mod 32 having $\sigma = 1$ is $d = 156$; the chains of reduced forms are (1, 12, -3, 12, 1; and its negative),

[8] And in fact, [1, 1, 7] is the only primitive reduced form with $d = -27$.

[9] If d is a square double these values for $\gamma(d)$.

$(2, 10, -7, 4, 5, 6, -6, 6, 5, 4, -7, 10, 2$; and its negative); by (8), these four classes each contain ambiguous forms of only one type.

4. The term *m-derived classes of C* will signify the primitive classes derived from C by integral transformations of determinant m.

For $f = [a, b, c]$ and $f' = [a, -b, c]$ it is plain from (13) that $fP_\kappa \sim f'P_{p-\kappa}$ $(0 < \kappa < p)$, $fP_0 \sim f'P_0$, $fP_p \sim f'P_p$. Hence by Lemmas 2 and 3 we have:

(a) *if K is a p-derived class of an ambiguous C, so is K^{-1};*

(b) *the p-derived classes of a non-ambiguous C are non-ambiguous, and are the reciprocals of the p-derived classes of C^{-1}.*

Both (a) and (b) extend at once to m-derived classes.

The m-derived classes of C are the same as the $(-m)$-derived classes of C^{-1}.

THEOREM 5. *Theorem 2 holds with p replaced by any non-zero integer m.*

5. If in (13), fP_p is primitive and p^2n is represented therein, evidently n is represented in f; similarly for fP_κ. By induction from m^2d to $(pm)^2d$: *an integral b. q. f. represents every number represented by any of its primitive derived classes* [Gauss, D. A., Art. 166].

Hence, if a primitive f_1 of discriminant m^2d represents a prime q (or $-q$ or ± 1) represented by f, then f_1 is a $(\pm m)$-derived form of f. For the property $f(q) > 0$ determines the class of f or its reciprocal.

THEOREM 6. *Let f and f_1 be primitive and of respective discriminants d and p^2d, any prime. Then f_1 is a $(\pm p)$-derived form of f if and only if f_1 represents p^2q, where $\pm q$ is any prime (or 1) represented by f, and if and only if (4) holds.*[10]

6. Now (9) is an immediate consequence of (4) and Theorem 3.

7. Since f represents all integers represented in f_1, each p-derived form f_1 of f has all the generic characters of f. An additional character

$$(f_1 \mid p),\ (-1 \mid f_1),\ (2 \mid f_1) \tag{30}$$

of f_1 occurs only in the following respective cases:

$$p > 2,\ p \nmid d;\ p = 2,\ d \equiv 4, 8\ (16);\ p = 2,\ d \equiv 16\ (32). \tag{31}$$

[10] Pall I, 331, II, 494.

LEMMA 6. *In the cases* (31), *among the* $p-(d \mid p)$ *primitive forms* (13) *occur equally many having either value* ± 1 *for* (30).

We may suppose a prime to p.

i) $p=2$. The forms (13) are now

(32) $f_0=[4a, 2b, c],\ f_1=[4a, 4a+2b, a+b+c],\ f_2=[a, 2b, 4c].$

We find that if $d \equiv 4$ or $8 \pmod{16}$, then $f_2 \equiv a \pmod 4$, and the primitive one of f_0, f_1 is $\equiv -a \pmod 4$; if $d \equiv 16 \pmod{32}$, then $f_2 \equiv a \pmod 8$ and the primitive one of f_0, f_1 is $\equiv 5a \pmod 8$.

ii) $p>2$. Now $\lambda = 2a\kappa + b$ is with κ an independent integral variable $(\mathrm{mod}\ p)$. If $f_1 = fP_p$, $(f_1 \mid p) = (a \mid p)$; and for the rest,

$$(f_1 \mid p) = (a \mid p)(\lambda^2 - d \mid p).$$

Hence Lemma 6 is a consequence of

LEMMA 7. *Let* m *be an integer,* p *an odd prime,* $p \nmid m$. *Then among the numbers*

$$\lambda^2 - m \qquad (\lambda = 0, 1, \cdots, p-1),$$

there is one more quadratic non-residue than residue $(\mathrm{mod}\ p)$.[11]

Consider cases (31). Each primitive genus Γ of discriminant d has two p-derived genera Γ_i $(i=1, -1)$ differing only in the value i of (30). Let $\Gamma(n)$ denote the number of sets of representations of n by a representative system of classes of Γ. By (4) and Theorem 3

(33) $$\Gamma_1(p^2n) = \Gamma_{-1}(p^2n) = \tfrac{1}{2}\{p-(d \mid p)\}\Gamma(n).$$

By (3), if m is prime to p, $\Gamma_i(pm) = 0$ $(i=1, -1)$; while

(34) $$\Gamma_i(m) = \Gamma(m),\ \Gamma_{-i}(m) = 0,$$
$$[i = (m \mid p), (-1 \mid m), (2 \mid m) \text{ according to (31)}].$$

There remain the cases (cf. (29))

(35) $p>2,\ p \mid d;\ p=2,\ d \equiv 1 \pmod 4,\ 12 \pmod{16},\ 0 \pmod{32}.$

The p-derived classes of Γ form a single genus Γ', for which, m and n being integers, $p \nmid m$,

[11] Thus there are $\theta = \tfrac{1}{2}\{p-(m \mid p)\}$ non-residues, $\theta - 1$ residues. This curious result must have appeared frequently in related forms.

$$(36) \qquad \Gamma'(p^2n) = \{p - (d \mid p)\}\Gamma(n),\ \Gamma'(pm) = 0,\ \Gamma'(m) = \Gamma(m).$$

By induction from (33) and (36_1) we have

THEOREM 7. *An integer n has equally many sets of representations in every primitive genus of discriminant d which represents n.*

To determine these genera we proceed as follows. To begin with, if n' has no prime factors p such that d'/p^2 is a discriminant, then $r(n', d') = 0$ unless

$$(37) \qquad p^a \mid n',\ p^{a+1} \nmid n',\ a \text{ odd imply } (d' \mid p) = 0 \text{ or } 1;$$

and then n' is represented in an unique genus Γ of discriminant d'.[12]

Generally, let k^2 be the largest square factor of n for which d/k^2 is a discriminant, and set $d = k^2d'$, $n = k^2n'$. Then n is represented only in the k-derived genera of the unique genus (if any) of discriminant d' which represents n'. The number of these genera, (37) being assumed, is 2^i, where i is the number of odd primes p such that $p \mid k$, $p \nmid d'$; this number being increased by 1 if

$$d' \equiv 1\ (4),\ k \equiv 4\ (8), \text{ or } d' \equiv 4\ (16),\ k \equiv 2\ (4); \text{ or}$$
$$d' \equiv 8,\ 16,\ 24\ (32),\ k \text{ even, or } d' \equiv 12\ (16),\ k \equiv 0 \pmod 4;$$

but increased by 2 if $d' \equiv 1 \pmod 4$, $k \equiv 0 \pmod 8$, or $d' \equiv 4\ (16)$, $k \equiv 0\ (4)$.

McGILL UNIVERSITY.

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[12] Cf. Pall I, 337, footnote. It is easy to define this genus by the generic characters either of n' or of the ambiguous forms representing the prime factors common to n' and d'.

ABSTRACT LINEAR DEPENDENCE AND LATTICES.

By GARRETT BIRKHOFF.

1. *Introduction.* In a preceding paper,[1] Hassler Whitney has shown that it is difficult to distinguish theoretically between the properties of linear dependence of ordinary vectors, and those of elements of a considerably wider class of systems, which he has called "matroids."

Now it is obviously impossible to incorporate all of the heterogeneous abstract systems which are constantly being invented, into a body of systematic theory, until they have been classified into two or three main species. The purpose of this note is to correlate matroids with abstract systems of a very common type,[2] which I have called "lattices."

As this correlation is purely formal, the discussion will be descriptive rather than detailed.

2. *The fundamental construction.* Let us refer to Whitney's definition of a matroid as a set M whose subsets have a numerical rank function with certain properties (R_1)-(R_3). It is easy to define mutual dependence by abstraction from the theory of vectors, and to see that if we suppress elements of rank zero, and regard mutually dependent elements as merely repeated occurrences of the same element, then we get a matroid M^* with the same structural properties as M. Moreover M^* has the additional property that no element is dependent on any other element.[3]

Now let us call a subset of M^* "linearly complete" if and only if it contains all elements dependent on it. By what we have just shown, distinct elements of M^* are distinct linearly complete subsets. Moreover

LEMMA 1. *The product $N \cdot N'$ of any two linearly complete subsets N and N' of M^*, is linearly complete.*

For suppose e any element of M^* not in $N \cdot N'$; by symmetry, we can suppose e not in N. Therefore

[1] H. Whitney, "On the abstract properties of linear dependence," *American Journal of Mathematics*, vol. 57 (1935), pp. 509-533. His main definition is in his section two.

[2] Cf. the author's "On the combination of subalgebras," *Proceedings of the Cambridge Philological Society*, vol. 39 (1933), pp. 441-464. This article will be referred to subsequently as "Subalgebras."

[3] This assertion corresponds to S. MacLane's assumption "without loss of generality" of his conditions (R_4)-(R_5) in his note on "Some interpretations of abstract linear dependence, etc.," *supra*.

$$\begin{aligned} 1 &= r(N+e) - r(N) && \text{[by definition of } N\text{]} \\ &\leqq r(N\cdot N'+e) - r(N\cdot N') && \text{[by Whitney's Lemma 4].} \end{aligned}$$

Hence $r(N\cdot N'+e) \geqq r(N\cdot N')+1$, and e is not dependent on $N\cdot N'$, proving the lemma.

Observe now that M^* is itself linearly complete, and has the property $M^*\cdot N = N$ for any subset N. Finally, the operation of intersection is idempotent, commutative, and associative. Consequently

THEOREM 1. *The linearly complete subsets of M^* can be regarded as the elements of a new system $L(M^*)$, satisfying*

(L_1) *Any a of $L(M^*)$ and b of $L(M^*)$ determine a unique "product" $a\cap b$ in $L(M^*)$.*

(L_2) *$a\cap a = a$, $a\cap b = b\cap a$, and $a\cap(b\cap c) = (a\cap b)\cap c$.*

(L_3) *There exists a "unit" i [namely, M^*] such that $a\cap i = a$ irrespective of a.*

(L_4) *In any sequence of products a_1, $a_1\cap a_2$, $a_1\cap a_2\cap a_3, \cdots$ some two terms are equal. [Remark: This is a simple corollary of the fact that $L(M^*)$ is finite].*

Moreover distinct single elements of M^* appear as distinct single elements of $L(M^*)$.

3. *An incidental theorem.* We shall now prove a result which is obvious in the case in hand—provided we define the "join" $N\cup N'$ of N and N' directly as the product of all linearly complete subsets containing N and N'. (This definition is equivalent to defining $N\cup N'$ as the set of all elements linearly dependent on $N+N'$). The formal proof *in abstracto* has however been included, to give a neat new approach to the theory of abstract lattices.[4]

THEOREM 2. *It is an abstract consequence of Theorem* 1 *that any elements a and b of $L(M^*)$ determine a unique "join" $a\cup b$ with the properties*

[4] It is extremely useful in applications. For instance, it shows that if we adjoin a purely formal "unity" i, we can subsume the theory of "effective implication" as defined by E. V. Huntington in his note "Effective equality and effective implication in formal logic," *Proceedings of the USA Academy of Sciences*, May, 1935, under the theory of lattices in the finite [but not the infinite!] case. Viewed in this light, his propositions 32, 33, 34, 36, 37 appear as extensions to the infinite case of Theorems 4. 2, 6. 1, 6. 2, 6. 3 of "Subalgebras."

With regard to precedence, it was Professor Huntington's paper which suggested Theorem 2 to me originally.

(L_5) $a \cap (a \cup b) = a \cup (a \cap b) = a$.

(L_6) $a \cup a = a$, $a \cup b = b \cup a$, *and* $a \cup (b \cup c) = (a \cup b) \cup c$.

The main thing is to define $a \cup b$. To do this, note first that $i \cap a = a$ and $i \cap b = b$. Second, not that if $c \cap a = a$ and $c' \cap a = a$, *then*

$$(c \cap c') \cap a = c \cap (c' \cap a) = c \cap a = a.$$

Therefore if we set $c_0 = i$, and try to find successive elements $c_1, c_2, c_3, \cdots$ such that $c_k \cap a = a$, $c_k \cap b = b$, and yet $c_0 \cap \cdots \cap c_k$ differs from every $c_0 \cap \cdots \cap c_j$ $[j < k]$, then by (L_4) we obtain after a finite number of attempts, an element $d = c_0 \cap \cdots \cap c_n$ such that (1) $d \cap a = a$ and $d \cap b = b$ (2) if $c^* \cap a = a$ and $c^* \cap b = b$, then $c^* \cap d = c_1 \cap \cdots \cap c_k$ for some $k < n$—which, since

$$c^* \cap d = c^* \cap (d \cap d) = (c^* \cap d) \cap d = c_1 \cap \cdots \cap c_k \cap d = d$$

[by iterated use of (L_2)], means $c^* \cap d = d$.

Conditions (1)-(2) and (L_2) show that d is determined uniquely by a and b; we shall *define* $a \cup b$ as d. To prove that $a \cup b$ satisfies (L_5)-(L_6) is now merely a question of substituting from (1)-(2), and using condition (L_2).

This completes the proof, the details of which the reader should be able to fill in without difficulty.

4. *Principal results.* But by definition, an "abstract lattice" is any system which satisfies (L_1)-(L_2) and (L_5)-(L_6). Consequently, we have

THEOREM 3. $L(M^*)$ *is an abstract lattice.*

Not all lattices correspond to matroids; in particular, every lattice corresponding to a matroid contains only a finite number of elements. The facts as to which lattices correspond to matroids can however be summarized as follows.

A finite lattice corresponds to a matroid if and only if (1) if b and c are smallest distinct elements larger than a, then $b \cup c$ is a smallest element larger than b. (This is the "dual" [under the inversion of inclusion [5]] of the abstract property of composition subgroups used in the classic proof of the Theorem of Jordan-Hölder) (2) every element can be expressed as the "join" of elements of rank one.

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[5] It being observed that the definition of a lattice is invariant under inversion of meet and join. Property (1) amounts in the case in hand to the property that if two linearly complete subsets of M^* of rank $(n+1)$ intersect in a subset of rank n, then the rank of their join is $(n+2)$.

Moreover the "rank" $r(N)$ of any subset N of elements $e_1, \cdots, e_k$ of M^* is equal to the length of any chain [in the sense of the Theorem of Jordan-Hölder] in $L(M^*)$, between the lattice-element corresponding to the null subset, and the "join" of the lattice-elements corresponding (under Theorem 1) to the e_i. That is,

THEOREM 4. *The structure of $L(M^*)$ determines that of M^*—and hence that of M.*

5. *Theoretical consequences.* The results of the last sections suggest trying to interpret theorems on matroids as theorems on lattices, and conversely. This can be done in at least two cases.

The less interesting of these is Whitney's Theorem 3, which may be regarded as a reappearance [6] of the first formula of Theorem 9.2 of "Subalgebras."

A more interesting correspondence concerns Whitney's Theorem 15 on separability, which is a generalization of a well-known theorem on graphs. This theorem may, in the light of the above results, be regarded as a specialization of a theorem [7] on the "strong" uniqueness of the representation of *any* lattice as the direct product of irreducible (= non-separable) components.

6. *Geometrical correspondence.* In another paper,[8] S. MacLane has shown that M^* also corresponds to a schematic configuration of dimensions $r(M^*) - 1$, and has pointed out a relationship with projective geometries. This defines an obvious correspondence *via* matroids between "matroid lattices" (or lattices satisfying conditions (1)-(2) of § 4) and schematic configurations.

In terms of this correspondence, results proved elsewhere [9] by the author directly imply

THEOREM 5. *A schematic configuration is the direct product of a finite number of projective geometries and single points, if and only if the dual [under the inversion of join and meet] of the corresponding matroid lattice, is again a matroid lattice.*

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[6] This formula very likely has a long previous history.

[7] Proved in the author's "On the lattice theory of ideals," *Bulletin of the American Mathematical Society*, vol. 40 (1934), p. 616, Theorem 2.

[8] Cf. footnote 3, above. A much weaker result of the same sort was obtained by the author, in Theorem 5 of "On the structure of abstract algebras," to appear shortly in the *Proceedings of the Cambridge Philological Society*.

[9] "Subalgebras," Theorem 10.2, and "Combinatorial relations in projective geometries," *Annals of Mathematics*, vol. 36 (1935), pp. 743-748.

(By the "direct product" of two projective geometries P and P^* is meant the system whose elements are the couples (A, A^*) [A any m-plane of P and A^* any m^*-plane of P^*], and in which (A, A^*) lies in (B, B^*) if and only if A lies in B and A^* in B^*. This is the inverse of the notion of separation into non-separable components mentioned above).

Theorem 5 and well-known results [10] show that such schematic configurations can in general be realized by vectors in spaces whose coördinates lie in suitable finite fields.

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HARVARD UNIVERSITY.

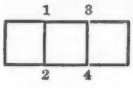
[10] O. Veblen and W. H. Bussey, "Finite projective geometries," *American Transactions*, vol. 7 (1906), pp. 241-259.

SOME UNIQUE SEPARATION THEOREMS FOR GRAPHS.[1]

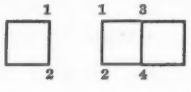
By SAUNDERS MACLANE.[2]

An abstract graph [3] is a combinatorial structure which consists of a finite set of points or *vertices,* together with a number of edges. Each edge *joins* exactly two of the vertices. Such graphs are considered in topology, in the study of electrical circuits [4] and in combinatorial analysis.[5] Since the structure of such a graph becomes involved if the number of vertices is large, it is desirable to have methods of uniquely breaking up a graph into smaller parts or components. One such method is that of separating the graph at a single vertex. This is possible when the removal of the vertex in question disconnects the graph. The uniqueness of this process has been established by Whitney.[6]

Similar to the separation at a single vertex is the separation along a chain of connected vertices. Consider for example the graph



The vertices 1 and 2 together form a chain. If we drop these two vertices and all edges involving them, the graph falls into two connected pieces. If the separating vertices 1 and 2 be added to each piece, we have the two components



The second component may be further separated by the chain (3, 4). This gives three components, each one a square. Had we instead separated by the chain (3, 4) first, the result would have been the same.

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[1] Presented to the Society, September 7, 1934.

[2] Sterling Research Fellow, Yale University.

[3] M. A. Sainte-Laguë, "Les Réseaux," *Mémorial des Sciences Mathématiques,* Fas. XVIII.

[4] R. M. Foster, "Geometrical Circuits of Electrical Networks," *Bell Telephone System Technical Publications,* Mon. B-653.

[5] E. Netto, *Lehrbuch der Combinatorik,* 2nd ed., pp. 292-308.

[6] H. Whitney, "Non-separable and Planar Graphs," *Transactions of the American Mathematical Society,* vol. 34 (1932), p. 339.

This uniqueness property can be suitably extended: if any graph is separated by chains until the resulting components cannot be further separated, then the result is independent of the chains used in this separation. Two provisos are needed: 1. all the chains used in the separation must be geodesics (§ 3); 2. isomorphic components are considered as essentially identical (§ 4).

The plan of the proof is as follows: it is sufficient to consider any two geodesic chains B and C, and to construct a separation S' beginning with a separation by B, and a separation T' beginning with C, in such a way that S' and T' have the same results. First we use the geodesic property to show that B and C form together a symmetric figure. We then construct the desired separation S' as an alternating separation; first separation by B, then by C, then by B, and so on. A slight modification is made to insure that each separation is a separation by a single connected chain. The so modified separation S' then has the same result as the separation by a similarly constructed T'. This is established by showing that each stage of the alternating separation S' gives a finer subdivision of the original graph than does the preceding stage of the separation T'.

The separation process has two other basic properties: it yields a representation of the original graph (§ 1) and it assists in the choice of complete independent sets of cycles (§ 2).

1. *The Definition of Separation.* The graphs as defined in the introduction will be restricted by requiring that each edge join two distinct vertices and that no pair of vertices be joined by more than one edge. We can then conveniently represent the vertices by arabic numerals, while each edge may be denoted by the pair of vertices which it joins. Two vertices joined by an edge are called *contiguous.* Any symmetric binary relation $R(x, y)$ in which x and y range over a finite class (the class of vertices) can be interpreted as a graph by taking R to be the relation of contiguity.

Among the *subgraphs* of a graph G we consider only those subgraphs H such that two vertices of H contiguous in G are also contiguous in H. This restriction is natural in the relational interpretation. Such a subgraph $H \subset G$ is completely determined by the set of its vertices.

If a graph G and two subgraphs H_1 and H_2 are given, then $H_1 + H_2$ denotes the subgraph determined by the set of all vertices present in either H_1 or H_2, while $H_1 - H_2$ denotes the subgraph determined by the set of vertices present in H_1 but not in H_2.

A graph with $n > 2$ distinct vertices $1, 2, \cdots, n$ and with the edges

$$(1\,2), (2\,3), \cdots, (n-1, n), (n, 1),$$

is called a *circuit*. Similarly, a *chain* is a graph of the type

$$(1\,2), (2\,3), \cdots, (n-1, n)$$

with $n > 0$ distinct vertices. This chain has the *length* n, and is said to *join* its *end* vertices 1 and n. Its vertices are ordered, in that a vertex j on the chain is *between* the vertices i and k if $i < j < k$ or $k < j < i$.

The separation of a connected graph G by a connected [7] subgraph H is an operation which yields a set of components, each of which is a subgraph of G. This set we denote by $H \times G$. To obtain it, dissect the graph $G - H$ into its connected pieces $F_1, F_2, \cdots, F_m$. The set of subgraphs

$$H_1 = F_1 + H, H_2 = F_2 + H, \cdots, H_m = F_m + H$$

is then the set $H \times G$. These components can also be characterized as follows:

THEOREM 1.1. *If H is a connected subgraph of a connected graph G, then every vertex of H belongs to every component of $H \times G$. Every vertex of $G - H$ belongs to just one component, and two such vertices belong to the same component if and only if they are joined in G by a chain containing no vertices of H.*

Any graph G is *represented* by the set of its components, in the following sense:

THEOREM 1.2. *If H is a connected subgraph of a connected graph G, then every vertex and edge of G is present in at least one component of $H \times G$.*

A set of components $H_1 \times G$ can be further separated by a subgraph H_2 of G if each component of $H_1 \times G$ which contains H_2 is separated by H_2, while the other components of $H_1 \times G$ are left unchanged. The result of this separation we denote by $H_2 \times H_1 \times G$. The simplest uniqueness theorem [8] then is

THEOREM 1.3. *If H_1 and H_2 are connected subgraphs of a connected graph G, and if H_1 and H_2 have no vertices in common, then*

$$H_1 \times H_2 \times G = H_2 \times H_1 \times G.$$

Proof. Denote the connected pieces of $G - (H_1 + H_2)$ by

[7] This assumption here is essential; without it any graph could be separated into trivial components.

[8] This theorem is essentially a generalization of Whitney's results for separation at single vertices.

$$G_1, G_2, \cdots, G_n.$$

A vertex b of $H_1 + H_2$ will be called a *boundary* vertex of a piece G_i if and only if b is contiguous to a vertex of G_i. Two applications of Theorem 1.1 then show that the operation $H_1 \times H_2 \times G$ yields a component

$$H_1 + H_2 + \Sigma' G_i, \tag{1}$$

the sum being taken over all these pieces G_i which have boundary vertices both in H_1 and in H_2, together with several components

$$H_1 + G_j, \tag{2}$$

one for each G_j whose boundary vertices are all in H_1, and several components

$$H_2 + G_k, \tag{3}$$

one for each G_k whose boundary vertices are all in H_2. Since these components, (1), (2), and (3), are symmetric in H_1 and H_2, the uniqueness is established.

This theorem enables us to define the separation of a connected graph G by a disconnected subgraph H. For if H consists of the connected pieces $H_1, H_2, \cdots, H_m$, we set

$$H \times G = H_1 \times H_2 \times \cdots \times H_m \times G.$$

2. *Circuits and Separation.* If we consider a graph G as a one-dimensional topological complex, then it is known that in this complex we can choose a *complete set* of linearly independent one-dimensional circuits.[9] Separation simplifies this choice, for the linearly independent circuits can all be chosen from the components *after* a separation $H \times G$.

To establish this result for a connected subgraph H, we agree to call a circuit C an *H-circuit* if $C - H$ is connected (in particular, C may be contained in H). An H-circuit C may contain no vertices of H whatsoever. If this is not the case, the vertices of C not in H form a chain with two ends a and b. The remaining vertices of C (i. e., those in H) form, together with a and b, a second chain with a and b as ends. Any H-circuit is thus contained in at least one component of $H \times G$.

LEMMA 2.1. *If a chain B in G contains no vertices of H, and if there is a circuit C containing B, then there is an H-circuit containing B.*

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[9] O. Veblen, *Analysis Situs*, 1st ed., p. 17.

Proof. If C is not itself an H-circuit, extend B along C until both end vertices are points of H. Then complete the desired circuit by joining these ends in H.

LEMMA 2.2. *Any circuit C in G may be represented as a sum (modulo 2) of H-circuits.*

Proof. Denote the connected pieces of $C - H$ by $B_1, B_2, \cdots, B_n$. Each B_i is a chain, and by Lemma 2.1 is embedded in an H-circuit C_i, so that $C_i - H = B_i$. The sum

$$C + C_1 + \cdots + C_n \text{ (modulo 2)}$$

has then no vertices not in H. It can thus be represented as a sum (modulo 2) of circuits in H; that is, as a sum of H-circuits. These circuits, with the C_i, give the representation [10] of C.

THEOREM 2.3. *If H is a connected subgraph of the connected graph G, then a complete set S of circuits in G can be constructed from $H \times G$ as follows: select in each component of $H \times G$ a complete set of circuits, combine all these sets to form a set T, and remove enough circuits from T to make the resulting set S linearly independent.*

Proof. The set is complete, for by Lemma 2.2 every circuit is a sum of H-circuits, while any H-circuit is contained in at least one component of $H \times G$. If H is a tree (i. e., contains no circuits), the set T itself can be shown to be linearly independent.

3. *Geodesic Separation.* The fundamental problem is to show that the result of separation by chains is essentially unique. This is not always true, for consider the graph which consists of three chains joining two vertices 1 and 2, with the edges

$$(1\,3),\ (3\,2),\ (1\,4),\ (4\,2),\ (1\,5),\ (5\,6),\ (6\,2).$$

If we separate this graph by the chain 1 3 2, we obtain a quadrilateral with vertices 1 2 3 4 and a pentagon with vertices 1 2 3 5 6. If instead we were to separate by the larger chain 1 5 6 2, we would obtain two pentagons, with vertices 1 2 3 5 6 and 1 2 4 5 6 respectively. The two results are essentially

[10] A circuit C contained (in the usual sense) in a graph G need not be a subgraph of G according to our definition; such a C can however always be represented as a sum (modulo 2) of circuits which are subgraphs.

different.[11] This difference arises because the two separating chains 1 3 2 and 1 5 6 2 have the same ends, but different lengths.

To avoid the lack of uniqueness we can agree in such a case to separate only by the shorter of the two chains; that is, by a "geodesic" chain. In general, a chain C is *geodesic* in a graph G if every other chain contained in G and having the same end vertices as C is at least as long as C. Note that any subchain of a geodesic chain is geodesic.

A *complete geodesic separation* of a graph G will then proceed first by separating G by a geodesic chain B, then by separating each resulting component L by some chain B_1, contained in and geodesic in L, and so on, until the final components are no longer separable in this way. The desired uniqueness theorem states that any two such complete geodesic separations S and T of the same graph G have essentially the same result. S will begin with a separation $B \times G$, while T starts with a separation $C \times G$. We first investigate the interrelations of the geodesics B and C.

This will involve the decomposition of a chain into subchains. A representation

$$D = D_1 + D_2 + \cdots + D_n$$

is said to be a *decomposition* of the chain D if each of the graphs D_i is a chain, if each vertex of D appears in one and only one of the chains D_i, and if the order of the sum on the right represents the order of the vertices of D. This last requirement means that if d_i, d_j, and d_k are vertices chosen from the three distinct chains D_i, D_j, and D_k, respectively, then d_j is between d_i and d_k in the chain D if and only if $i < j < k$ or $k < j < i$.

LEMMA 3.1. *Let B and C be geodesic chains in a graph G, and let f, g, and h be vertices belonging to both B and C. If g is between f and h in B, then g is also between f and h in C.*

Proof. Since g lies between f and h, there is a decomposition

$$B = B_1 + f + B_2 + g + B_3 + h + B_4,$$

in which f (likewise g and h) represents the subchain whose only vertex is f. If f were between g and h in C, then similarly

$$C = C_1 + g + C_2 + f + C_3 + h + C_4.$$

We thus have two distinct chains

[11] The results cannot be further reduced, for a circuit is not separable by any subgraph.

$$f + B_2 + g + B_3 + h$$
$$f + C_3 + h$$

joining the same vertices f and h. Both chains are geodesic, and hence have the same length. If the length of each chain B_i (or C_i) be denoted by the corresponding Greek letter, then

$$\beta_2 + 1 + \beta_3 = \gamma_3.$$

A similar argument for the chains joining g to h yields

$$\gamma_2 + 1 + \gamma_3 = \beta_3.$$

Substitution of the second equation in the first gives the contradiction

$$(\beta_2 + \gamma_2 + 2) + \gamma_3 = \gamma_3.$$

The intersection of two geodesic chains can now be described as follows:

THEOREM 3. 2. *Let B and C be geodesic chains in a graph G, and let the intersection of B and C consist of exactly n connected pieces $A_1, \cdots, A_n$. Then these A_k are chains, and there are two decompositions*

$$B = B_1 + A_1 + B_2 + A_2 + \cdots + B_n + A_n + B_{n+1},$$
$$C = C_1 + A_1 + C_2 + A_2 + \cdots + C_n + A_n + C_{n+1},$$

where the B_i and C_i are chains, no two of which have a vertex in common. Each B_j and each C_j for $j = 2, \cdots, n$ contains at least one vertex. The orientation of the A_k in both decompositions is the same; that is, we can denote the ends of each A_k by a_k and e_k in such a way that a_k is contiguous to a vertex of B_k and a vertex of C_k, while e_k is similarly contiguous to B_{k+1} and C_{k+1}. However, a_1 need not be contiguous to a void B_1, etc. Furthermore

$$\text{length } B_j = \text{length } C_j \quad (j = 2, \cdots, n)$$

Finally, a vertex d on B_j $(j = 2, \cdots, n)$ is never contiguous to any vertex of C_i $(i \neq j)$.

Proof. By suitably numbering the pieces A_k we can obtain the decomposition of B. A similar decomposition for C is possible, except that the chains A_k might appear here in a different order. But since each A_k contains at least one vertex, any such difference is impossible, by Lemma 3. 1.

Furthermore, the decomposition of C might offer some A_k in an orientation different from that in the decomposition of B. Thus A_k might have ends

a_k and e_k, with a_k contiguous to B_k and C_{k+1} and e_k contiguous to B_{k+1} and C_k. We can assume without loss of generality that $k < n$, and we select a point d in A_n. We then have e_k between a_k and d in B, while a_k is between e_k and d in C, contrary to Lemma 3. 1.

Next, the equality of the lengths follows because

$$e_{j-1} + B_j + a_j, \qquad e_{j-1} + C_j + a_j$$

are two geodesic chains with the same ends. Finally, the non-contiguity of B_j and C_i is a consequence of the geodesic property of C.

The chains A_k $(k = 1, \cdots, n)$, B_i, and C_i $(i = 1, \cdots, n+1)$ into which B and C have thus been decomposed may be called the *atomic* chains of the figure. No two atomic chains have vertices in common.

4. *Isomorphic Components.* Two complete geodesic separations of a graph need not give identical results. Consider for instance the graph used in § 3. The separation by 132 gave a quadrilateral 1234 and a pentagon 1 2 3 5 6. The separation by the geodesic chain 1 4 2 would give the same quadrilateral, but a different pentagon 1 2 4 5 6. The two different pentagons are nevertheless still isomorphic.

In general, two graphs are said to be isomorphic [12] if their vertices can be put into a 1 — 1 correspondence which leaves invariant the relation of contiguity. It is important to note the effect of separation on isomorphic graphs.

THEOREM 4. 1. *If two connected graphs G_1 and G_2 are isomorphic, and if G_1 be separated by a subgraph H_1, while G_2 is separated by the corresponding subgraph H_2, then there is a 1 — 1 correspondence between the components of $H_1 \times G_1$ and those of $H_2 \times G_2$, such that corresponding components are isomorphic.*

This theorem is a consequence of the general invariance of any logical property under an isomorphism.

5. *Alternating Separation.* Let a graph G and two geodesic chains B and C, with intersections as described in § 3, be given. We proceed to construct a separation S', beginning with $B \times G$, and a separation T', beginning with $C \times G$, such that S' and T' have the same results. S' is to be constructed in stages of alternating separations by B and by C, as indicated in

$$(1) \qquad \cdots \; C \times B \times C \times B \times G,$$

[12] Whitney, "On the Classification of Graphs," *American Journal of Mathematics*, vol. 55 (1933), p. 236.

and the first m stages of the separation will be denoted by $(C, B)^m \times G$. T' is to have a similar structure. However, the simple alternation indicated in (1) is not always sufficient.[18] Thus it may happen that $B \times G$ yields a component L which does not contain all of C, such that the part of C in L consists of two separate chains D_1 and D_2. Then instead of next separating L by C (or, more precisely, by the disconnected graph $D_1 + D_2$), it is desirable to separate L by a graph C', obtained by adding to $D_1 + D_2$ enough vertices from B to insure that the result C' is a single chain. Similar modifications are to be made at each stage of the alternating separation.

This modified separation $(C, B)^m \times G$ will be defined by induction on m. At the same time we will establish the following properties of any component L of $(C, B)^m \times G$:

5. 1. If one vertex of an atomic chain (see § 3, end) D belongs to L then D is contained in L;

5. 2. For every j, $j = 2, \cdots, n$, either B_j or C_j belongs to L;

5. 3. For every k, $k = 1, \cdots, n$, A_k belongs to L.

For $m = 1$ we define

$$(C, B)^1 \times G = B \times G.$$

The three properties follow at once from Theorem 1. 1.

Now let m be even and equal to $2p$, and let L be a component of the already defined separation $(C, B)^{2p-1} \times G$. L is to be separated by the chain C, after C has been modified as follows:

1. If C_1 is not present in L, drop C_1 from C;
2. If C_{n+1} is not present in L, drop C_{n+1} from C;
3. If C_j is not present in L $(j = 2, \cdots, n)$, replace C_j in C by the corresponding B_j.

These transformations of C yield a graph C_L, with the following properties:

5. 4. C_L is a chain,

5. 5. C_L is contained in L,

5. 6. C_L contains every vertex of C present in L,

5. 7. C_L is geodesic.

The first property is a consequence of the configuration of B and C, as described in Theorem 3. 2. The construction of C_L, together with properties

[18] Thus if G has nine vertices and the edges (1 2), (2 3), (3 4), (2 5), (5 4), (1 6), (6 7), (7 4), (1 8), (8 9), (9 4), and if B and C are the chains 1 2 3 4 and 2 5 4 respectively, the simple alternating separations S' and T' will not give the same results.

5. 1 and 5. 2, gives the second and third properties. Finally, to show that C_L is geodesic, note that the first and second operations above transform C into a subchain, and thus leave it a geodesic, while, in virtue of Theorem 3. 2, the third operation changes neither the length nor the ends of C, and so must leave it geodesic.[14] C_L is thus geodesic in G, and therefore geodesic in L as well.

$(C, B)^{2p} \times G$ is now the set of all components obtained by separating each component L of $(C, B)^{2p-1} \times G$ by the corresponding chain C_L. The three properties 5. 1, 5. 2, and 5. 3 of the separation are established by using Theorem 1. 1 and noting that the separator C_L contains every A_k and just one of each pair B_j, C_j. A component M of $(C, B)^{2p} \times G$ which arises from a component L of $(C, B)^{2p-1} \times G$ by the separation $C_L \times L$ is said to have L as a *parent*. Every such component M has at least[15] one parent.

The inductive definition of the alternating separation may be completed for the case of odd m by a treatment similar to that for even m, except that the letters B and C are to be everywhere interchanged. Thus each component M of $(C, B)^{2p} \times G$ is to be separated by a suitable modification B_M of the chain B. In any component L at any stage of separation $(C, B)^m \times G$ both the separators B_L and C_L can be constructed, although only one is to be used.

The phenomenon of isomorphic but not identical results discussed in § 4 can affect but a small part of a component L; that part of L unaffected in this way we will call the *body* of L. Specifically, the body of L consists of all vertices of L, except that B_j is to be omitted from the body if C_j is not present in L, while C_j is to be omitted if B_j is not present $(j = 2, \cdots, n)$. We will denote the body of L by $\beta(L)$. A vertex not in $\beta(L)$ belongs to both B_L and C_L.

The "finer separation theorem" to be discussed in the next section applies only[16] to those components L with $B_L \neq L$ and $C_L \neq L$. Such components we call *substantial*. A substantial component L contains at least one vertex not in B_L and one vertex not in C_L. These vertices are useful in establishing uniqueness properties.

6. *Substantial Components in Alternating Separation.* The ultimate

[14] C would not necessarily remain a geodesic were C_1 to be replaced by B_1. For this reason, the end chains B_1 and B_{n+1} cannot be treated in the same way as the intermediate chains $B_2, \cdots, B_n$.

[15] Two or more parents for one component can occur; see § 7.

[16] For example, if G has six vertices and the edges (1 2), (2 3), (3 4), (2 5), (5 4), (1 6), (2 6), while $B = 2\,5\,4$ and $C = 1\,2\,3\,4$, then $(B, C)^2 \times G$ and $(C, B)^2 \times G$ differ in that the former has a non-substantial component 1 2 5 4 not present in the latter.

agreement of the two separations $(C, B)^m \times G$ and $(B, C)^m \times G$ rests on the increasing fineness of the separation. Thus $C \times B \times G$, since it involves separations by both C and B, should subdivide G more finely than $C \times G$ alone. This "finer separation theorem" is

THEOREM 6.1. *If L is a substantial component of $(C, B)^m \times G$ $(m > 0)$, then there exists a component M of $(B, C)^{m-1} \times G$, such that*

1. $\beta(L) \subset \beta(M)$,
2. *M is substantial.*

First note that the following conclusions could be added to those of the theorem.

3. If $g \in L$ and $g \notin C_L$ for some vertex g, then $g \in M$ and $g \notin C_M$.
4. The same for B_L and B_M.

These follow readily from the first conclusion. By the definition of the body of L, $g \in \beta(L)$. Hence $g \in \beta(M)$ and $g \in M$. Were $g \in C_M$, then either $g \in C$, whence by 5.6 $g \in C_L$, a contradiction, or $g \in B_j$ for some $j = 2, \cdots, n$ such that C_j is absent from M. In the latter case $g \notin \beta(M)$, a contradiction.

The second conclusion of the theorem follows from the third conclusion, in virtue of the definition of substantial components.

The first conclusion itself may be proved by induction on m. If $m = 1$, we need only take $M = G$. If $m > 1$, we treat the typical case of even m. We select a parent L' of the given component L. The induction assumption yields a component M' in $(B, C)^{m-2} \times G$, with

$$\beta(L') \subset \beta(M').$$

The desired component M is now to be found as a component of $C_{M'} \times M'$ corresponding to L. Since L is substantial, there is a vertex g in L but not in C_L. But L has arisen by the separation $C_{L'} \times L'$, whence

$$C_L = C_{L'}. \tag{1}$$

Conclusion 3. above gives $g \in M'$, $g \notin C_{M'}$. Hence there is exactly one component M of $C_{M'} \times M'$ which contains g. This M, a component of $(B, C)^{m-1} \times G$, has the desired property 1, for if $h \in \beta(L)$, we prove $h \in \beta(M)$ by an argument in two cases, as follows.

Case 1. $h \notin C_L$. Then, by (1), $h \notin C_{L'}$. But since the vertex g chosen above to determine M lies together with h in the component L of $C_{L'} \times L'$.

there is by Theorem 1.1 some chain D in L' which joins g to h and contains no vertices of $C_{L'}$. By the induction assumption, $D \subset M'$ and D contains no vertices of $C_{M'}$. Hence g and h must belong to the same component M of $C_{M'} \times M'$. Finally, $h \in M$ and $h \not\in C_{M'}$ imply $h \in \beta(M)$.

Case 2. $h \in C_L$. It may be that h is in C_1 or C_{n+1} or some A_k. The desired conclusion $h \in \beta(M)$ is then immediate. Otherwise, since $h \in \beta(L)$, there is some j, $(j = 2, \cdots, n)$ such that $h \in C_j$, while both C_j and B_j are contained in L. Hence both C_j and B_j are contained in L', in $\beta(L')$, and therefore, by the induction assumption, in $\beta(M')$. By 5.6 $C_j \subset C_{M'}$, and hence $C_j \subset M$ in the separation $C_{M'} \times M'$, while by case 1 $B_j \subset M$. Since both B_j and C_j are in M, C_j and therefore h is in $\beta(M)$.

It is also necessary to show that the alternating separation used ultimately comes to an end. This is the finiteness theorem.

THEOREM 6.2. *For a given graph G and fixed geodesic chains B and C there exists an integer q such that, for all $m > q$,*

$$(B, C)^m \times G = (B, C)^q \times G.$$

We thus introduce the notation for an "infinite" separation:

$$(B, C)^\infty \times G = (B, C)^q \times G.$$

Proof. Any component L in a separation $(B, C)^m \times G$ is obtained from some parent component M by a separation such as $B_M \times M$. Hence $B_M \subset L$ and $B_L = B_M$, so that L cannot be further separated by the same separating chain B_L. If now L should remain unchanged by the *next* separation $C_L \times L$, it must remain unchanged by *every* subsequent separation, for then neither B_L nor C_L separates L. That is, if L is separated at any one of the subsequent stages of separation, then it must also be actually separated at the next stage.

Return to the theorem and take q equal to the number of vertices in G. Suppose that the inequality

$$(B, C)^{m+1} \times G \neq (B, C)^m \times G$$

holds for some $m \geqq q$. Then there is at least one component L_0 on the left different from one of its parents L_1 on the right. Denoting one of the parents of L_1 by L_2, and of L_2 by L_3, and so on, construct the sequence

$$L_0, L_1, L_2, \cdots, L_q.$$

Since $L_0 = L$, the remarks of the preceding paragraph show that each L_i is different from its parent L_{i+1}; in particular, each L_i has a smaller number of vertices than its parent. Since L_q cannot have more than q vertices, this shows that L_0 has no vertices at all, a contradiction.[17]

Any separation $H \times G$ is determinate, in that a vertex of G not in H is contained in exactly one component of $H \times G$. A useful consequence is

THEOREM 6.3. *If L and N are two substantial components of $(C, B)^m \times G$, such that*

$$\beta(L) \subset \beta(N),$$

then

$$L = N.$$

Proof. If L contains a vertex f contained in neither B nor C, then, by the determinateness mentioned above, L is the only component of $(C, B)^m \times G$ containing f. Since $f \in \beta(L)$, the result follows.

Suppose that no such vertex f is present. L consists then entirely of atomic chains from B and C. Since L is substantial, it contains either a pair of chains (B_j, C_j) for some $j = 2, \cdots, n$, or else one of the pairs (B_1, C_1), (B_1, C_{n+1}), (B_{n+1}, C_1), (B_{n+1}, C_{n+1}). Now there is but one component of $(C, B)^m \times G$ which contains both B_j and C_j, as may be established from the determinateness property by induction. Since, by the definition of the body, both B_j and C_j belong to $\beta(L)$ and thus to $\beta(N)$ if they belong to L, the equality $L = N$ is again established. The other pairs above are treated in like manner.

The uniqueness theorem for substantial components now runs as follows:

THEOREM 6.4. *There is a $1-1$ correspondence between the substantial components L of $(C, B)^\infty \times G$, and the substantial components M of $(B, C)^\infty \times G$, such that corresponding components are isomorphic.*

A suitable correspondence between the L's and the M's can be defined by the relation

$$\beta(L) = \beta(M). \tag{1}$$

This relation is equivalent to the relation

$$\beta(L) \subset \beta(M), \tag{2}$$

[17] The value of q used here can be decreased considerably by a more elaborate analysis, based on the number n of chains common to B and C.

for, by the finer separation Theorem 6.1 there exists for each M an N in $(C, B)^\infty \times G$ with

$$\beta(M) \subset \beta(N).$$

This, combined with (2), shows that the conditions of Theorem 6.3 hold. The conclusion (1) follows at once. The converse, (1) implies (2), is immediate.

Furthermore, (1) is a $1-1$ relation. The existence of an M for every L follows from the finer separation Theorem 6.1 and the equivalence of (2) and (1). The uniqueness of this M is a consequence of Theorem 6.3. The converse properties are established by similar proofs.

If L and M stand in the relation (1), then they are isomorphic. For (1), combined with 5.2 and the definition of a body of a component, shows that L and M differ, if at all, only in that, for certain values of j between 2 and n inclusive, L contains B_j but not C_j, while M contains C_j but not B_j (or vice versa). If in L we replace each such B_j by the corresponding C_j, or vice versa, we can transform L into M.

Each replacement $B_j \to C_j$ in this transformation is an isomorphism. Since B_j and C_j have the same number of vertices, we need only show that the contiguity relations of B_j and those of C_j are the same. In fact, both B_j and C_j are attached to the adjacent chains A_{j-1} and A_j in the same manner (see Theorem 3.2). No other contiguities between vertices of B_j and vertices of $L - B_j$ are possible. For suppose instead that a vertex g of B_j were so contiguous to a vertex h of L. This vertex is not in B, for B is a chain. If h were in C but not in B, then either $h \in C_j$, and both C_j and B_j belong to L, contrary to assumption, or $h \in C_i$ for $i \neq j$, contrary to Theorem 3.2. Hence h belongs to neither B nor C, and so $h \in \beta(L)$. By (2), $h \in \beta(M)$. Now in any separation $H \times G$ two contiguous vertices such as h and g must be present together in at least one component; in the separation $(B, C)^\infty \times G$ this must be the component M, for h is in only one component. But $g \in M$ contradicts the assumption that C_j but not B_j was contained in M. The invariance of the contiguity is thus established.[18]

7. *Duplicate and Non-substantial Components.* In a complete geodesic separation S a component may consist of just one edge e. However, the

[18] The isomorphism of L to M is of a special type: any vertex g in L is identical with its corresponding vertex in M unless g is not contiguous to more than two points in L (alternatively, L is transformed into M by replacing certain "suspended" chains). The whole investigation could be formulated in terms of this narrower concept of isomorphism.

same edge e may be part of another component of S. In such a case we agree to drop the edge e from the set of components S. On the other hand, if e appears more than once as a component in S, but is not part of any larger component, then we agree to count e but once. This convention [19] will not alter the fundamental Theorems 1. 2 and 2. 3.

More generally, any set W of components may contain a component L which is a "duplicate" because it is a subgraph of some other component in W. In a chain separation all such duplicate components are chains; hence the result of a complete geodesic separation is not altered if all duplicates are dropped as they arise.

The non-substantial components in a complete separation are either duplicates or arise from "bridges." By a *bridge* [20] e in a graph G we mean an edge which is contained in no circuit of G.

THEOREM 7. 1. *If S is a complete geodesic separation of a connected graph G, then every bridge in G appears by itself as a component in S, and S contains no other components which are trees.*

Proof. A tree with more than one edge is always separable; hence the only trees in S are single edges of G. Any edge e of G which is not a bridge is contained in a circuit of G. For any separation $B \times G$, e is contained in a B-circuit (§ 2), and this circuit is contained in some component L of $B \times G$. Hence e belongs to a component which is not a tree. By induction, e is present in S in some component which is not a tree. Hence if e appears by itself in S, it is a duplicate.

Any bridge e' of G belongs to at least one component L in S and is a bridge in L. Were L to contain any edges other than e', then L would be separable, contrary to the fact that S is a complete separation. Hence e' is the whole component L.

This theorem suggests that the treatment of duplicate components in a complete separation can be standardized as follows: Whenever a component L with a bridge e arises, then separate L successively by the two end-vertices of e. Then e appears by itself as a component. It is a duplicate unless it was a bridge in the original graph G.

8. *Uniqueness Theorems for Chain Separation.*

THEOREM 8. 1. *Any two complete geodesic separations S and T of a connected graph G have the same result, in that there is a* 1 — 1 *correspond-*

[19] Some such convention is necessary for the uniqueness theorems of § 8.

[20] E. Steinitz und H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, p. 98.

ence between the components of S and the components of T such that corresponding components are isomorphic.

This theorem may be proved by induction on the number q of vertices in G. An enumeration of the cases shows it to be true for $q = 1, 2, 3, 4$. Assume then that it holds for all integers less than some q. Let the separations S and T begin with $B \times G$ and $C \times G$ respectively, and construct a complete separation S' beginning with $(C, B)^\infty \times G$. Also construct a complete separation T' beginning with $(B, C)^\infty \times G$ and continuing so that each substantial component of $(B, C)^\infty \times G$ is separated by T' in just the same way as the corresponding isomorphic component of $(C, B)^\infty \times G$ is separated by S'. By Theorem 6.4, $(C, B)^\infty \times G$ and $(B, C)^\infty \times G$ have the same result, except possibly for some components which are non-substantial; i. e., which are chains. Thence, by Theorems 4.1 and 7.1, S' and T' have the same result throughout. Now S and S' both begin with the separation $B \times G$. We can assume without loss of generality that $B \times G \neq G$; we can then apply the induction assumption to each component of $B \times G$ and obtain a $1-1$ correspondence between components of S and S' which are not single edges. The correspondence can then be extended to include these single edges (Theorem 7.1). By a similar proof, T and T' have the same result. Hence S, through S' and T', has the same result as does T.

A similar theorem can be obtained for a complete separation which uses geodesic chains whose lengths are all less than a fixed integer n_0. This follows because all the separating chains used in the construction of $(B, C)^\infty \times G$ are at least as short as the longer of the two chains B and C.

For example, a chain of length 2 is always a geodesic. Hence if a graph is separated by single vertices and by edges (chains of length 2) as many times as possible, then the resulting set of components is independent of the particular vertices and edges used in the separation. In this case corresponding components are not only isomorphic, but also identical, for two chains B and C of length 2 cannot have more than one subchain in common (see Theorem 3.2 and the discussion of isomorphic components under Theorem 6.4).

The methods discussed in this paper for establishing the uniqueness of the separation of graphs by chains can be applied to study separation by circuits. The chief problem which arises is the development of a notion of "geodesic" circuits to exclude the cases in which separation is not unique.

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GAUSSIAN DISTRIBUTIONS AND CONVERGENT INFINITE CONVOLUTIONS.

By AUREL WINTNER.

A result of Paley and Zygmund [1] [3] on Rademacher series suggests the following situation: *Every symmetric Bernoulli convolution is at the infinity at least as strongly damped as a Gaussian distribution function.* The object of the first part of the present note is the proof of a theorem to this effect. The result implies in particular that every symmetric Bernoulli convolution has *moments* of arbitrarily high order,

$$(1) \qquad \mu_m(\phi) = \int_{-\infty}^{+\infty} x^m \, d\phi(x), \qquad (m = 0, 1, 2, \cdots),$$

and that these moments belong to a *determined* moment problem. As far as the asymptotic estimate by means of Gaussian distributions is concerned, it cannot be stated that the *density* of probability is majorized by a Gaussian density $C \exp(-\lambda x^2)$. In fact, there exist symmetric Bernoulli convolutions which are *nowhere* absolutely continuous (cf. [2], Example 6). It will, however, be shown that *the distribution function itself* is

$$(2) \qquad O(\omega_\lambda(x)) \text{ as } x \to -\infty \text{ and } 1 + O(1 - \omega_\lambda(x)) \text{ as } x \to +\infty,$$

where

$$\omega_\lambda(x) = (\lambda/\pi)^{1/2} \int_{-\infty}^{x} \exp(-\lambda u^2) \, du$$

is the Gaussian distribution function with a sufficiently small dispersion parameter $\lambda > 0$. Since there need not exist a density, the situation is more delicate than in the case of "smooth" problems of the type of the distribution problem of the Riemann ζ-function (cf. [2], Theorem 16). It will turn out that *it is not necessary to restrict the considerations to the case of Bernoulli convolutions.* It may be emphasized that the Gaussian distribution function, majorizing the *convergent* infinite convolutions under consideration, belongs to a case of *divergence* of these infinite convolutions (cf. the end of the paper). The proof of (2) requires an adaptation to convolutions of an argument applied by Paley and Zygmund [3] to Rademacher series. The adaptation is made possible by the well-known algorithm of convolution moments (cf., e. g.,

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[4]). This algorithm will play the rôle of the algorithm which defines the orthogonal functions of Rademacher. The result (2) for symmetric Bernoulli convolutions is interesting also in view of a theorem of Hardy [1], since it has been shown in [5] that there exists for every $\epsilon > 0$ a symmetric Bernoulli convolution the Fourier transform of which is

$$O(\exp(-|t|^{2-\epsilon}))$$

as $t \to \pm\infty$.

Let

$$\sigma_1, \sigma_2, \cdots, \sigma_n, \cdots \tag{3}$$

be a sequence of distribution functions which are, for instance, constant without a finite interval. On considering the Fourier transform of the convolution ("Faltung")

$$\sigma_1 * \cdots * \sigma_n$$

and using the notation (1), it is easily shown (cf. [4]) that

$$\mu_m(\sigma_1 * \cdots * \sigma_n) = [\mu(\sigma_1) + \cdots + \mu(\sigma_n)]^m,$$

where $[\ ]^m$ is a symbolical power affecting the μ with subscripts. In other words,

$$\mu_m(\sigma_1 * \cdots * \sigma_n) = \sum C_{ij\ldots}\mu_i(\sigma_1)\mu_j(\sigma_2)\cdots, \tag{4}$$

where the summation runs through all collections $(i, j, \cdots)$ of n non-negative integers $i, j, \cdots$ for which $i + j + \cdots = m$, and $C_{ij\ldots}$ denotes the multinomial coefficient

$$C_{ij\ldots} = (i + j + \cdots)!\, i!^{-1} j!^{-1} \cdots.$$

It is easy to see (cf. [3]) that

$$C_{2i\,2j\ldots} \leqq m^m C_{ij\ldots}, \quad \text{if} \quad i + j + \cdots = m. \tag{5}$$

Let, in particular,

$$\mu_{2m+1}(\sigma_1) = 0,\ \mu_{2m+1}(\sigma_2) = 0, \cdots, \mu_{2m+1}(\sigma_n) = 0, \cdots, \tag{6}$$

where $m = 0, 1, 2, \cdots$. Then from (4)

$$\mu_{2m+1}(\sigma_1 * \cdots * \sigma_n) = 0,$$

while

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) = \sum C_{2i\,2j\ldots}\mu_{2i}(\sigma_1)\mu_{2j}(\sigma_2)\cdots,$$

where the summation runs through all collections $(i, j, \cdots)$ for which $2i + 2j + \cdots = 2m$. Hence from (5).

$$(7)\quad \mu_{2m}(\sigma_1 * \cdots * \sigma_n) \leqq m^m \Sigma C_{ij\ldots}\mu_{2i}(\sigma_1)\mu_{2j}(\sigma_2)\cdots, \quad (m = 1, 2, \cdots),$$

where the summation runs through all collections for which $i + j + \cdots = m$.

Let

$$(8)\qquad \sigma_n(x) = \beta(x/b_n), \qquad (b_n > 0,\ n = 1, 2, \cdots),$$

where $\beta(x)$ denotes the symmetric Bernoulli distribution function which is equal to 0, $\frac{1}{2}$ or 1 according as x lies on the left, in the interior or on the right of the interval $-1 \leqq x \leqq 1$. Then (6) is satisfied, while

$$\mu_{2m}(\sigma_n) = {b_n}^{2m} \qquad (m = 0, 1, 2, \cdots;\ n = 1, 2, \cdots).$$

Hence from (7)

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) \leqq m^m \Sigma C_{ij\ldots} {b_1}^{2i}{b_2}^{2j}\cdots,$$

where the summation runs through all collections of n non-negative integers $i, j, \cdots$ for which $i + j + \cdots = m$ and so the factor Σ of m^m is the m-th power of ${b_1}^2 + {b_2}^2 + \cdots + {b_n}^2$. Consequently,

$$(9)\qquad \mu_{2m}(\sigma_1 * \cdots * \sigma_n) < (Bm)^m, \qquad (m, n = 1, 2, \cdots)$$

where

$$B = \sum_{n=1}^{\infty} {b_n}^2.$$

It will be supposed that this series is convergent. This assumption is, according to [2], necessary and sufficient for the existence of a distribution function τ such that

$$(10)\qquad \sigma_1 * \cdots * \sigma_n \to \tau \text{ as } n \to +\infty,$$

i. e., for the convergence of the infinite convolution

$$(10a)\qquad \tau = \sigma_1 * \sigma_2 * \cdots,$$

where σ_n is defined by (8). Now it is easy to see that $\mu_{2m}(\tau)$ is finite for every m and that, furthermore,

$$(11)\qquad (0 \leqq)\mu_{2m}(\tau) \leqq (Bm)^m, \qquad (m = 1, 2, \cdots).$$

In fact, it is clear from (9) and (1) that, for every $R > 0$,

$$\int_{-R}^{R} x^{2m}\, d\tau_n(x) < (Bm)^m, \text{ where } \tau_n = \sigma_1 * \cdots * \sigma_n.$$

Since, from (10),

$$\int_{-R}^{R} x^{2m}\, d\tau_n(x) \to \int_{-R}^{R} x^{2m}\, d\tau(x), \qquad n \to +\infty,$$

in virtue of Helly's theorem on term-by-term integration, it follows that

$$\int_{-R}^{R} x^{2m}\, d\tau(x) \leqq (Bm)^m,$$

which implies (11) by letting $R \to +\infty$.

The estimate (11) shows that the series

$$\sum_{m=1}^{\infty} [\mu_{2m}(\tau)]^{-1/(2m)}$$

is divergent. Hence

$$\mu_m(\tau) = c_m \qquad (m = 0, 1, 2, \cdots),$$

is a determined moment problem in virtue of Carleman's classical criterion. It is also seen from (11) and from the convergence of the series

$$\sum_{m=1}^{\infty} m!^{-1} \lambda^m B^m m^m, \text{ where } m!^{-1} m^m \sim (2\pi m)^{-1/2} e^m,$$

for sufficiently small values of $\lambda > 0$ that the series

$$\sum_{m=0}^{\infty} m!^{-1} \lambda^m \int_{-\infty}^{+\infty} x^{2m} d\tau(x) = \sum_{m=0}^{\infty} m!^{-1} \lambda^m \mu_{2m}(\tau)$$

is convergent, and so

$$\int_{-\infty}^{+\infty} \exp(\lambda x^2)\, d\tau(x) < +\infty$$

for every sufficiently small $\lambda > 0$. Thus for every $x > 0$

$$\text{const.} > \int_{x}^{+\infty} \exp(\lambda u^2)\, d\tau(u) \geqq \exp(\lambda x^2) \int_{x}^{+\infty} d\tau(u) = \exp(\lambda x^2)[1 - \tau(x)],$$

or

$$1 - \tau(x) = O(\exp(-\lambda x^2)), \qquad x \to +\infty,$$

while

$$\tau(x) = O(\exp(-\lambda x^2)), \qquad x \to -\infty$$

by reasons of symmetry. This completes the proof of (2), since λ is arbitrary and

$$\exp((\lambda - \epsilon)x^2) \int_{x}^{+\infty} \exp(-\lambda u^2)\, du < \int_{x}^{+\infty} \exp((\lambda - \epsilon)u^2 - \lambda u^2)\, du$$
$$= \int_{x}^{+\infty} \exp(-\epsilon u^2)\, du \to 0, \qquad 0 < x \to +\infty;$$

hence

$$\int_x^{+\infty} \exp(-\lambda u^2)\,du = O(\exp(-(\lambda-\epsilon)x^2)), \qquad x \to +\infty,$$

for every fixed positive ϵ $(<\lambda)$.

The above proof applies also in cases where (10a) is not a Bernoulli convolution. In fact, *it is sufficient to assume that* (10a) *is a convergent infinite convolution of arbitrary symmetric distribution functions* σ_n *for which*

$$B < +\infty, \quad \textit{where} \quad B = \sum_{n=1}^{\infty} \mu_2(\sigma_n),$$

and

$$(12) \qquad (\mu_2(\sigma_n))^m \geqq \alpha^m \mu_{2m}(\sigma_n), \qquad (m = 0, 1, 2, \cdots;\ n = 1, 2, \cdots),$$

where α *is a sufficiently small positive constant.*

First, the symmetry of σ_n means that

$$\sigma_n(x) + \sigma_n(-x) = 1.$$

Since this implies (6), the inequality (7) is applicable. Hence, from (5) and (12),

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) \leqq m^m \alpha^{-m} \Big(\sum_{j=1}^{n} \mu_2(\sigma_j) \Big)^m,$$

and so

$$(9a) \qquad \mu_{2m}(\sigma_1 * \cdots * \sigma_n) \leqq (Am)^m, \qquad (m, n = 1, 2, \cdots),$$

where

$$A = \alpha^{-1} B \geqq \alpha^{-1} \sum_{j=1}^{n} \mu_2(\sigma_j)$$

is independent of m and n. The balance of the proof is the same as above.

A sufficient condition for (12) is that

$$(8a) \qquad \sigma_n(x) = \rho(x/b_n), \qquad (b_n > 0,\ n = 1, 2, \cdots),$$

where $\rho(x)$ is an arbitrary symmetric distribution function the spectrum of which is a bounded set and contains at least two points. In fact, the latter conditions imply that

$$|\mu_m(\rho)|^{1/m} < \text{const. and } \mu_2(\rho) \neq 0.$$

On the other hand,

$$\mu_m(\sigma_n) = b_n{}^m \mu_m(\rho)$$

by the definition of σ_n. Thus (12) is satisfied. The convergence of

$$B = \sum_{n=1}^{\infty} b_n{}^2$$

is necessary and sufficient for the convergence of the infinite convolution (10a) also when $\rho \neq \beta$. On placing

$$x = y\sqrt{B_n}, \text{ where } B_n = b_1^2 + \cdots + b_n^2,$$

the finite convolution

$$\sigma_1(x) * \cdots * \sigma_n(x)$$

tends, as $n \to +\infty$, to a Gaussian distribution function $\omega = \omega(y)$ whenever $B = +\infty$ and $b_n < \text{const.}$, while it tends to the distribution function $\tau(y\sqrt{B})$ whenever $B < +\infty$, i. e., whenever (10a) is a convergent infinite convolution.

THE JOHNS HOPKINS UNIVERSITY.

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ON CONVERGENT POISSON CONVOLUTIONS.

By AUREL WINTNER.

Let

$$\pi = \pi(x) = \pi(x; a, q), \qquad -\infty < x < +\infty,$$

where

$$a > 0 \quad \text{and} \quad 0 < q < 1,$$

denote the one-dimensional distribution function which has at $x = 0$ and $x = a$ the jumps $1 - q$ and q respectively. Thus $\pi(x)$ is a step-function which is equal to

$$0, \quad 1 - q \quad \text{or} \quad 1,$$

according as x is in the interval

$$(-\infty, 0), \quad (0, a) \quad \text{or} \quad (a, +\infty).$$

Clearly

(1) $$\mu_k(\pi) = a^k q, \quad \text{hence} \quad \mu_2(\pi) - [\mu_1(\pi)]^2 = a^2 q(1-q),$$

and

(2) $$L(t; \pi) = 1 - q(1 - e^{iat}),$$

where $\mu_k(\sigma)$ denotes the k-th moment

(1a) $$\mu_k(\sigma) = \int_{-\infty}^{+\infty} x^k \, d\sigma(x), \qquad (k = 1, 2, \cdots; \ \mu_0(\sigma) = 1),$$

and $L(t; \sigma)$ the Fourier transform

(2a) $$L(t; \sigma) = \int_{-\infty}^{+\infty} e^{itx} \, d\sigma(x), \qquad (-\infty < t < +\infty),$$

of the distribution function σ. Poisson's law of seldom events deals, in its classical form [1] [2], with the limit of a properly reduced convolution of a finite number of distribution functions π. The present note considers infinite convolutions of distinct distribution functions π without any reduction, that is to say infinite convolutions

(3) $$\pi_1(x) * \pi_2(x) * \pi_3(x) * \cdots * \pi_n(x) * \cdots,$$

where

(4) $$\pi_n(x) = \pi(x; a_n, q_n),$$

and the asterisk is the convolution operator.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

It has been shown in [1] that if both series

$$\sum_{n=1}^{\infty} \mu_1(\sigma_n), \qquad \sum_{n=1}^{\infty} \{\mu_2(\sigma_n) - [\mu_1(\sigma_n)]^2\} \tag{5}$$

are convergent, then so is the infinite convolution

$$\sigma_1(x) * \sigma_2(x) * \cdots. \tag{6}$$

Now the series (5) take in the case (3) the forms

$$\sum_{n=1}^{\infty} a_n q_n, \qquad \sum_{n=1}^{\infty} a_n^2 q_n (1 - q_n) \tag{7}$$

in virtue of (4) and (1), and it is shown by examples like

$$q_n \sim n^{-4}, \qquad a_n^2 q_n (1 - q_n) = 1 \qquad (a_n q_n \sim n^{-2},\ n \to \infty) \tag{8}$$

that the convergence of the first series (7) is compatible with the divergence of the second series (7). It is nevertheless true that the convergence of the first series (7) alone implies the convergence of (3). It is sufficient to prove that *the convergence of the first series* (5) *alone implies the convergence, even the absolute convergence, of* (6), *whenever no* $\sigma_n(x)$ *has a negative* x *in its spectrum.* Now in the latter case

$$\mu_1(\sigma_n) = \int_{-\infty}^{+\infty} |x| \, d\sigma_n(x),$$

and it has been shown in [8] that the assumption

$$\sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} |x| \, d\sigma_n(x) < +\infty$$

always implies the absolute convergence of (6). Hence the situation is this:

The infinite Poisson convolution (3) *is absolutely convergent whenever*

$$\sum_{n=1}^{\infty} a_n q_n < +\infty. \tag{9}$$

On the other hand, (9) is not implied by the absolute convergence of (3); cf. (2) and (14b).

It will always be supposed that (9) is satisfied. The distribution function represented by (3) will be denoted by

$$\rho = \rho(x). \tag{3a}$$

Thus

$$L(t;\rho) = \prod_{n=1}^{\infty} L(t;\pi_n) \tag{10}$$

according to the general theory [1] of infinite convolutions. It is seen from (8) that (9) may be satisfied also when

$$a_n \to +\infty \quad \text{as} \quad n \to +\infty. \tag{11}$$

This does not contradict the fact, proven in [1], that the convergence of (3) implies the limit relation

$$\pi_n \to \delta, \tag{9a}$$

where δ denotes the distribution function

$$\delta(x) = \tfrac{1}{2}(1 + \operatorname{sgn} x). \tag{12}$$

In fact, the n-th term of the series (9) is the area of the rectangle which is bordered by the curves

$$y = \pi_n(x) \quad \text{and} \quad y = \delta(x),$$

and so (9a) is implied by the condition

$$a_n q_n \to 0, \tag{9b}$$

which is compatible with (11).

In view of Lévy's deduction [4] of the Gauss law, it is worth while to mention that, as a consequence of the convergence criterion (9), *there exist infinite convolutions such that*

$$\mu_1(\sigma_n) = 0 \quad \textit{and} \quad \mu_2(\sigma_n) = 1 \quad \textit{for every } n, \tag{13}$$

although (6) *is convergent.* If (6) is convergent, then the finite convolution

$$\sigma_1(x) * \sigma_2(x) * \cdots * \sigma_n(x),$$

when reduced to have the dispersion 1, does not tend to the Gaussian distribution function of dispersion 1 but to the distribution function (12), for which $\mu_2(\delta) = 0$. Now the Fourier transforms of the distribution functions

$$\sigma(x) \quad \text{and} \quad \sigma(x+b) \tag{13a}$$

are identical up to the factor e^{itb}. Hence, if

$$\sum_{n=1}^{\infty} b_n \tag{13b}$$

is convergent, the convergence of (6) is necessary and sufficient for the convergence of the "translated" infinite convolution

$$\sigma_1(x+b_1) * \sigma_2(x+b_2) * \cdots .$$

Thus the existence of a convergent infinite convolution (6) satisfying (13) is, in view of (1), (9) and (13b), a consequence of the fact tha. (8) is compatible with (9), the difference of the first moments μ_1 of the distribution functions (13a) being equal to b in virtue of (1a).

The addition rule of spectra [1], when applied to (3), shows that the spectrum of (3) always consists of $x=0$ and of the closure of the positive values

$$a_i + a_j + a_k + \cdots , \tag{14}$$

where $i, j, k, \cdots$ is an arbitrary finite collection of distinct positive integers. Thus the spectrum of (3) is a bounded set if and only if

$$\sum_{n=1}^{\infty} a_n < +\infty , \tag{14a}$$

a condition implying (9). The smoothness criterion (21), to be proven later on, is compatible with (14a). On the other hand, (3) cannot be a continuous function if

$$\sum_{n=1}^{\infty} q_n < +\infty . \tag{14b}$$

In fact, it follows by a complete induction from the definition of a convolution as an integral, that the jump of the function (3) at $x=0$ is never less than

$$\prod_{n=1}^{\infty} (1-q_n),$$

and is therefore positive in the case (14b). Further, it is clear from the spectrum rule (14) that the spectrum of (3) is the whole interval $[0, +\infty]$, whenever

$$\sum_{n=1}^{\infty} a_n = +\infty \quad \text{and} \quad a_n \to 0. \tag{14c}$$

This does not imply the continuity of (3), since (14c) is compatible with (14b) and (9). An example to this effect is

$$q_n = a_n^2 = (n+1)^{-2}.$$

Now if both (14b) and (14c) are satisfied, then the discontinuity points of

(3) lie dense in $[0, +\infty]$ in virtue of Theorem 35 of [1]. The spectrum of (3) is a sequence of points

$$0, c_1, c_2, \cdots, \text{ where } c_n \to +\infty, \tag{15}$$

if, for instance, (11) is satisfied or if a_n is independent of n, conditions necessitating (14b) in view of (9). On the other hand, not every distribution function which has a spectrum of the structure (15) may be represented in the form (3). An example to this effect is the Poisson distribution function [2], for which $c_n = n$ and the jump at $x = n$ $(\geqq 0)$ is $(n!\, e)^{-1}$.

An approach to the investigation of smoothness properties of $\rho(x)$ consists of obtaining, if possible, estimates of $L(t;\rho)$ for $t \to \pm\infty$; cf. [7]. In this direction it will first be shown that if

$$\sum_{n=1}^{\infty} q_n^2 < +\infty, \tag{16}$$

where

$$q_n \leqq \tfrac{1}{3}, \tag{16a}$$

and if (9) is satisfied, then

$$\log |L(t;\rho)| < -Ct^2 f(t) + A, \tag{17}$$

where C and A are positive constants and $f(t)$ denotes the even function

$$f(t) = \sum_{a_n |t| \leqq 1} q_n a_n^2. \tag{18}$$

The inequality (17) is understood to hold for every t, with the proviso that $\log |L|$ denotes $-\infty$ if $L = 0$.

The proof of (17) depends on a combined adaptation of arguments used in [7] and [9]. First,

$$\log |L(t;\pi)| < -q(1-\cos at) + 8q^2, \quad \text{if} \quad q \leqq \tfrac{1}{3}. \tag{19}$$

In fact,

$$\log |L(t;\pi)| + q(1-\cos at)$$

is the real part of the function

$$\log L(t;\pi) + q(1 - e^{iat})$$

and so not larger than the absolute value of this function, hence, according to (2),

$$\leqq \sum_{n=2}^{\infty} |q(1-e^{iat})|^n/n \leqq \sum_{n=2}^{\infty} (2q)^n/n < 8q^2,$$

if $q \leqq \frac{1}{3}$. This proves the inequality (19), which is, according to (16a), applicable for every $q = q_n$. Now

$$1 - \cos t \geqq Ct^2$$

for some $C > 0$ and for every t in the interval $-1 \leqq t \leqq 1$. Hence

$$1 - \cos a_n t \geqq Ca_n^2 t^2$$

for every $a_n \leqq 1/|t|$. Consequently,

$$\sum_{a_n|t|\leqq 1} q_n(1 - \cos a_n t) \geqq \sum_{a_n|t|\leqq 1} q_n C a_n^2 t^2$$

for every t. On combining this inequality with (18) and (19) and using the assumption (16), it is seen that

$$\sum_{a_n|t|\leqq 1} \log |L(t;\pi_n)| < -Ct^2 f(t) + A$$

for some sufficiently large $A > 0$. This implies

$$\sum_{n=1}^{\infty} \log |L(t;\pi_n)| < -Ct^2 f(t) + A$$

in virtue of $|L(t;\pi_n)| \leqq 1$ and therefore completes, according to (10), the proof of (17).

Since the spectrum of $\rho(x)$ lies at the right of the origin, $\rho(x) = 0$ for $x < 0$ but not for every x, so that $\rho(x)$ cannot be regular-analytic along the x-axis. On the other hand, it is possible to delimit a class of sequences (4) for which $\rho(x)$ has derivatives of arbitrarily high order for every x. According to [7], this will be the case whenever

$$L(t;\rho) = O(\exp -|t|^{\gamma}), \qquad t \to \pm\infty, \tag{20}$$

for some $\gamma > 0$. Now, on using (17), it is easy to prove that *if*

$$\tfrac{1}{2} < \nu < 1 < \lambda + \nu \tag{21}$$

and if (4) *is such that*

$$n^{-\delta-\lambda} < a_n < n^{\delta-\lambda} \quad \text{and} \quad n^{-\delta-\nu} < q_n < n^{\delta-\nu} \tag{22}$$

for every $\delta > 0$ *and every* $n > N = N_\delta$, *then* $\rho(x)$ *has for every* x *derivatives of arbitrarily high order,* (20) *being satisfied for every*

$$\gamma < (1-\nu)/\lambda. \tag{23}$$

At the first sight, the restrictions (21) seem to be artificial. It turns out, however, that *the inequalities* (21) *are about the best of their kind.*

First, (9) and (16) are satisfied in virtue of (21) and (22). Also, (16a) is implied by (16) for large n, and since in (23) the sign of equality is anyway excluded, no generality is lost by supposing that (16a) holds not only for large n but for every n. Thus (17) is applicable. Now from (21)

$$2\lambda + \nu > 1;$$

hence, as $|t| \to \infty$,

$$\sum_{n^{-\lambda}|t|\leq 1} n^{-2\lambda-\nu} \sim \int_{|t|^{1/\lambda}}^{\infty} u^{-2\lambda-\nu} du = B\,|t|^{(1-2\lambda-\nu)/\lambda}, \tag{24}$$

where $B > 0$. Since δ in (22) is arbitrarily small, it is seen from (21), (22) and (24) by straight-forward estimates that

$$\sum_{a_n|t|\leq 1} a_n^2 q_n > |t|^{(1-2\lambda-\nu-\epsilon)/\lambda}$$

for every $\epsilon > 0$ and for every $|t| > T = T_\epsilon$. Hence, from (18) and (17),

$$\log|L(t;\rho)| < -|t|^{(1-\nu-\epsilon)/\lambda}$$

for every $\epsilon > 0$ and for every $|t| > T' = T'_\epsilon$. This means that (20) is true for every γ satisfying (23), q. e. d.

That the inequalities (21) are, in the main, as good as possible, is seen as follows. First, $\nu > 1$ is impossible for any λ, since $\nu > 1$ implies (14b), hence a discontinuity for $\rho(x)$. On the other hand, on letting

$$2\nu \to 1 + 0 \quad \text{and} \quad 2\lambda \to 1 + 0,$$

which is compatible with (21), it is seen that

$$(1-\nu)/\lambda \to 1 - 0,$$

while (20) holds for every γ satisfying (23). Hence (21) permits in (20) *every* $\gamma < 1$. Now (20) cannot hold in the limiting case $\gamma = 1$. In fact, (20) with $\gamma = 1$ implies, as shown in [7], the analyticity of $\rho(x)$ along the x-axis, when, as a fact, $\rho(x)$ cannot be analytic along the x-axis. This proves that $\nu < \frac{1}{2}$ is impossible for some λ. The assumption $1 < \lambda + \nu$ has been made in view of (9). Finally, it was necessary to exclude in (21) the equality signs, since one had to take care of the small exponents $\pm\delta$ in (22).

These considerations hold, of course, also under the restriction $\lambda = \nu$, in which case (21) goes over into $\frac{1}{2} < \nu < 1$ and the rectangle of small area, mentioned in connection with (12), is nearly a square.

Without using the language of the convolution theory, Schoenberg has effectively shown [5] that the distribution problem of the sequence

$$(25a)\quad \log 1 - \log \phi(1),\ \log 2 - \log \phi(2), \cdots, \log n - \log \phi(n), \cdots$$

depends on the particular case

$$(25)\qquad q_n = 1/p_n \sim 1/(n \log n), \qquad a_n = -\log (1 - 1/p_n) \sim q_n$$

of the infinite convolutions treated in the present note, $\phi(n)$ being Euler's function and p_n the n-th prime number. Schoenberg obtains (10) from a direct consideration of Schur and not from the theory of convolutions. An application of the theory of convolutions to arithmetical functions which are more general than (25a) will be given by Schoenberg in a paper referred to in [1]. It is seen from (14c) that in the case (25) the spectrum of $\rho(x)$ is the whole interval $[0, +\infty]$. On the other hand, Schoenberg [5] has proven in an interesting manner that $\rho(x)$ has, in the case (25), no discontinuity points. It is not known whether Schoenberg's $\rho(x)$ is or is not absolutely continuous; the criterion (20) fails, since (25) implies $\nu = \lambda = 1$. Correspondingly, Schoenberg's proof for the continuity of $\rho(x)$ depends on arithmetical properties of his a_n, more particularly on their linear independence, and not merely on questions (22) of rough order of magnitude.

It is interesting that the existence of a rough order of magnitude implies a high degree of smoothness for the symmetric case of infinite "Bernoulli convolutions" also. The result of [7] is precisely to this effect but imposes a strong restriction of regularity on the increase of the correction factor. The method used above admits of a simpler and general treatment if one replaces (17) by the corresponding inequality given in [9].

In order to prove this, let $\beta = \beta(x)$ be the distribution function which has both at $x = -1$ and $x = 1$ the jump $\frac{1}{2}$, i. e., the distribution function for which $L(t;\beta) = \cos t$, and let the sequence $\{b_n\}$ be such that

$$(26)\qquad \sum_{n=1}^{\infty} b_n{}^2 < +\infty, \quad \text{where} \quad b_n > 0.$$

Then the infinite Bernoulli convolution

$$(27)\qquad \sigma(x) = \beta(x/b_1) * \beta(x/b_2) * \cdots$$

is convergent and has the Fourier transform

$$(28)\qquad L(t;\sigma) = \prod_{n=1}^{\infty} \cos (b_n t).$$

On placing

$$N(t) = \sum_{b_n|t|\geqq 1} 1, \tag{29}$$

it is easy to see that

$$L(t;\sigma) = O(\exp\{-C[N(2t) - N(t)]\}) \qquad (t \to \pm\infty) \tag{30}$$

for some $C > 0$. In fact, $N(t)$ clearly is the number of values n satisfying the inequality $b_n \geqq 1/|t|$, where $b_n > 0$. Hence, if $t > 0$, there are at least

$$N(2t) - N(t) \tag{29a}$$

values of n satisfying both inequalities

$$1/(2t) \leqq b_n \leqq 1/t. \tag{29b}$$

Now for all these (29a) values of n,

$$0 < \cos(b_n t) \leqq \cos\tfrac{1}{2} \tag{29c}$$

in virtue of (29b). Hence it is seen from (28) that

$$|L(t;\sigma)| \leqq (\cos\tfrac{1}{2})^{N(2t)-N(t)} \tag{30a}$$

for every $t > 0$. Since $L(t;\sigma)$ and $N(t)$ are even functions in view of (28) and (29), the estimate (30) is implied by (30a), the constant C being the logarithm of $1/\cos\frac{1}{2}$, and so $C > 0$. Now (30) implies the desired analogue to (22), and even more:

If (26) *is satisfied and if*

$$n^{-\delta-\alpha} < b_n < n^{\delta-\alpha} \tag{31}$$

holds for a fixed $\alpha > 0$, *for every* $\delta > 0$ *and for every* $n > K = K_\delta$, *then* (27) *has derivatives of arbitrarily high order for every* x. *Furthermore,* (27) *is or is not regular analytic along the* x*-axis according as* $\alpha < 1$ *or* $\alpha > 1$. *On the other hand,* (27) *may possess derivatives of arbitrarily high order for every* x *also in cases where the decrease of* b_n *is stronger than that allowed by* (31), *an example to this effect being*

$$b_n = \exp(-n^\kappa) \qquad (0 < \kappa < \tfrac{1}{2}). \tag{32}$$

Finally, in the case

$$b_n = c^n, \qquad c = 2^{-1/h} \qquad (h = 1, 2, \cdots), \tag{33}$$

where the decrease of b_n *is still stronger,* (27) *acquires derivatives of arbitrarily high order for every* x *as*

$$c \to 1 - 0, \text{ i. e., } h \to + \infty,$$

for a fixed h there being assured the existence of a continuous $(h-1)$-th derivative for every x.

The last result is interesting also because (27) is, according to [3], not an absolutely continuous function if $0 < c < \frac{1}{2}$ (for $c = \frac{1}{3}$ one obtains the classical Cantor function).

Consider first (31). Since (31) holds for every $\delta > 0$, it is easily seen from (29) and (31) that

$$| t |^{(1-\epsilon)/\alpha} < N(t) < | t |^{(1+\epsilon)/\alpha}$$

for every $\epsilon > 0$ and for every $| t | > T = T_\epsilon$. This clearly implies

$$N(2t) - N(t) > | t |^{(1-\eta)/\alpha}$$

for every $\eta > 0$ and for every $| t | > T' = T'_\eta$. It therefore follows from (30) that

$$L(t;\sigma) = O(\exp(- | t |^\gamma))$$

for every $\gamma < 1/\alpha$. Thus there always exists a $\gamma > 0$, and $\gamma = 1$ is admissible whenever $\alpha < 1$. Finally, if $\alpha > 1$, then

$$\sum_{n=1}^{\infty} b_n < + \infty,$$

i. e., the spectrum of (27) is a bounded set and so (27) cannot be regular analytic along the x-axis. The spectrum of the simplest case, $b_n = n^{-\alpha}$, has further been discussed in [3].

Consider next (32). The function $N(t)$ is, according to (29) and (32), the largest integer not exceeding $(\log | t |)^{1/\kappa}$, where $| t | > 1$. A straightforward reduction yields therefore for large values of $| t |$ the asymptotic formula

$$N(2t) - N(t) \sim (\kappa^{-1} \log 2)(\log | t |)^{(1-\kappa)/\kappa}.$$

Thus, since $0 < \kappa < \frac{1}{2}$,

$$N(2t) - N(t) > \epsilon^{-1} \log | t |$$

for every $\epsilon > 0$ and for every $| t | > T = T_\epsilon$. Hence from (30)

$$L(t;\sigma) = O(| t |^{-1/\epsilon})$$

for every fixed $\epsilon > 0$, an estimate implying the existence of all derivatives of (27) for every x. In fact, it is seen from the inversion formula [4],

$$\sigma(x) = \sigma(0) + \int_{-\infty}^{+\infty} (2\pi i t)^{-1}(1 - e^{-itx}) L(t;\sigma)\, dt, \quad \left(\int_{-\infty}^{+\infty} = \lim \int_{-T}^{T}\right),$$

of the Fourier transform (2a) that, for a fixed $h > 0$, the estimate

$$L(t;\sigma) = O(|\, t\,|^{-h}) \qquad (t \to \pm\infty) \tag{34}$$

is a sufficient condition for the existence of continuous derivatives $\sigma^{(k)}(x)$ of every order $k \leqq h - 1$.

The remaining case (33) is now easily treated. In fact, from (28) and (33),

$$L(t;\sigma) = \prod_{n=1}^{\infty} \cos(2^{-n/h} t).$$

Hence

$$L(t;\sigma) = \prod_{j=0}^{h-1} \prod_{m=1}^{\infty} \cos(2^{-m}\, 2^{-j/h} t). \tag{35}$$

This may be written, in virtue of Vieta's identity

$$\prod_{m=1}^{\infty} \cos\,(2^{-m} z) = z^{-1} \sin z,$$

in the form

$$L(t;\sigma) = \prod_{j=0}^{h-1} (2^{-j/h} t)^{-1} \sin(2^{-j/h} t).$$

Consequently,

$$L(t;\sigma) = \prod_{j=0}^{h-1} O(|\, t\,|^{-1}) = O(|\, t\,|^{-h}),$$

which completes the proof in view of the criterion (34).

It may be mentioned that $\sigma(x)$ is, in the case (33), *symmetrically convex*, i. e., such that the derivative $\sigma'(x)$ is a non-increasing function of $|\, x\,|$. In fact, if two distribution functions are symmetrically convex, then so is their convolution. Now on denoting by $\sigma_h(x)$ the $\sigma(x)$ which belongs to (33), it is seen from (35) that

$$L(t;\sigma_h) = \prod_{j=0}^{h-1} L(2^{-j/h} t;\sigma_1),$$

i. e.,

$$\sigma_h(x) = \sigma_1(x) * \sigma_1(2^{1/h} x) * \cdots * \sigma_1(2^{(h-1)/h} x).$$

Thus it is sufficient to show that $\sigma_1(x)$ is symmetrically convex. Now $L(t;\sigma_1)$ is Vieta's product and so

$$L(t;\sigma_1) = t^{-1} \sin t = \tfrac{1}{2} \int_{-1}^{1} e^{itx}\, dx;$$

hence

$$\sigma'_1(x) = \tfrac{1}{2} \quad \text{if} \quad |x| < 1 \quad \text{and} \quad \sigma'_1(x) = 0 \quad \text{if} \quad |x| > 1.$$

Thus $\sigma_1(x)$ is symmetrically convex.

Similarly, the $\sigma(x)$ belonging to $b_n = n^{-1}$ is symmetrically convex, since it may be written in the form

$$\sigma(x) = \sigma_1(x) * \sigma_1(3x) * \sigma_1(5x) * \cdots.$$

This $\sigma(x)$ is regular for every x, since in (31) the correction factors $n^{\pm\delta}$ are superfluous and so

$$L(t;\sigma) = O(\exp(-|t|^\gamma))$$

holds not only for every $\gamma < 1/\alpha = 1$ but for $\gamma = 1$ also.

THE JOHNS HOPKINS UNIVERSITY.

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A NOTE ON THE CONVERGENCE OF INFINITE CONVOLUTIONS.

By AUREL WINTNER.

It is known[1] that the infinite convolution $\sigma_1 * \sigma_2 * \cdots$ of distribution functions $\sigma_n = \sigma_n(x)$ is convergent if and only if the product

$$(1) \qquad \prod_{n=1}^{\infty} L(t;\sigma_n), \quad \text{where} \quad L(t;\sigma_n) = \int_{-\infty}^{+\infty} e^{itx}\, d\sigma_n(x),$$

is uniformly convergent in every fixed finite t-interval. This criterion for the convergence of $\sigma_1 * \sigma_2 * \cdots$ has the disadvantage of involving a variable parameter t. Convergence criteria which contain only the first and second moments of every σ_n have been proven in the paper referred to before, where a general theory of $\sigma_1 * \sigma_2 * \cdots$ also is developed. The object of the present note is to point out a simple convergence criterion which does not contain a parameter and applies also when the moments of second or first order do not exist. Let $\delta > 0$ and

$$(2) \qquad \sum_{n=1}^{\infty} M_\delta(\sigma_n) < +\infty, \quad \text{where} \quad M_\delta(\sigma_n) = \int_{-\infty}^{+\infty} |x|^\delta\, d\sigma_n(x).$$

If (2) holds for $\delta = \delta_0$, it need not hold for $\delta < \delta_0$, although it requires the less of every σ_n the smaller is δ. *Now if* $0 < \delta \leqq 1$, *then* (2) *implies the absolute convergence of* $\sigma_1 * \sigma_2 * \cdots$. For if $0 < \delta \leqq 1$, then $|\sin tx| \leqq |tx|^\delta$. Now $|e^{itx} - 1| \leqq 2\,|\sin(tx/2)|$. Hence $|e^{itx} - 1| \leqq 2\,|tx|^\delta$. Consequently, since $L(0;\sigma_n) = 1$,

$$|L(t;\sigma_n) - 1| \leqq \int_{-\infty}^{+\infty} 2\,|tx|^\delta\, d\sigma_n(x) = 2\,|t|^\delta M_\delta(\sigma_n).$$

Hence it is seen from (2) that the product (1) is absolutely and uniformly convergent in every fixed finite t-interval. *Furthermore,* (2) *implies the convergence of* $\sigma_1 * \sigma_2 * \cdots$ *only when* $\delta \leqq 1$. For let $\sigma_n(x) = 0$ or $\sigma_n(x) = 1$ according as $x \leqq 0$ or $x \geqq 2/n$, while $\sigma_n(x) = 1/2$ if $0 < x < 2/n$. Then $2M_\delta(\sigma_n) = (2/n)^\delta$, hence (2) holds for every $\delta > 1$, although

$$\prod_{n=1}^{\infty} L(t;\sigma_n) = \prod_{n=1}^{\infty} \exp(it/n)\cos(t/n)$$

is not convergent for every t.

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DETERMINATION OF A VAN DER CORPUT-LANDAU ABSOLUTE CONSTANT.

By RICHARD KERSHNER.

It has been shown by van der Corput and Landau [1] that there exists an absolute constant γ with the property that if $f(x)$ be a real-valued function possessing in the finite interval $[a, b]$ a second derivative which is nowhere less than a fixed positive constant r, then

$$\left| \int_a^b \cos f(x)\, dx \right| \leqq \gamma r^{-\frac{1}{2}}, \tag{1}$$

it being understood that γ is independent of $[a, b]$ as well as of $f(x)$. This result has several applications in the analytic theory of numbers. Let γ_0 denote the least permissible value γ. The best estimate of γ_0 so far known seems to be

$$\gamma_0 \leqq 32^{\frac{1}{2}} = 5.657 \cdots.$$

We shall determine the actual value of γ_0 by verifying a suggestion of Wintner that the maximum of the expression on the left of (1) occurs when $f(x)$ is a parabola such that $f''(x) \equiv r$. The precise result is included in the following statement:

Let $f(x)$ be real-valued and possess a second derivative $f''(x) \geqq r > 0$ in $[a, b]$. Then

$$\left| \int_a^b \cos f(x)\, dx \right| \leqq \int_{-\beta_0}^{\beta_0} \cos(rx^2/2 + \mu_0)\, dx = \gamma_0 r^{-\frac{1}{2}},$$

where

$$\beta_0 = [(\pi - 2\mu_0)/r]^{\frac{1}{2}},$$

the constant [2] $\mu_0 = -0.725 \cdots$ *being determined as the only root of*

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[1] Cf. E. Landau, *Vorlesungen über Zahlentheorie*, vol. 2 (1927), p. 60; or E. Landau, *Einführung in die Differentialrechnung und Integralrechnung* (1934), p. 307; also for further references E. C. Titchmarsh, "On van der Corput's method and the zeta function of Riemann," *Quarterly Journal of Mathematics*, vol. 2 (1932), p. 173.

[2] The values of μ_0 and γ_0 given here were obtained by interpolation into the table of Fresnel integrals included in G. N. Watson, *A Treatise on the Theory of Bessel Functions* (1922), and are correct to $\pm .002$.

$$\int_0^{(\pi/2-\mu)^{1/2}} \sin(x^2+\mu)\,dx = 0$$

in $-\pi/2 \leqq \mu \leqq \pi/2$, *so that*

$$\gamma_0 = 2\sqrt{2}\int_0^{(\pi/2-\mu_0)^{1/2}} \cos(x^2+\mu_0)\,dx = 3.327\cdots.$$

It will be sufficiently clear from the proof that the function $f(x) = rx^2/2 + \mu_0$ and the interval $[a, b] = [-\beta_0, \beta_0]$ lead, for any fixed r, to a maximum for the expression on the left of (1) which is unique up to the obviously permissible translations which do not affect the value of $\cos f(x)$.

The method of van der Corput and Landau consists, in the main, in introducing $f(x)$ as an independent variable and applying the second mean value theorem. In the present paper we shall work directly with elementary properties of the cosine curve. In fact, the proof follows directly from the following lemma.

LEMMA. *Let* $f_1(x)$, $f_2(x)$ *be real-valued and possess first derivatives* $f'_1(x)$, $f'_2(x)$ *in the intervals* $[a, b_1]$, $[a, b_2]$ *respectively.*

Suppose that

(i) $$f_1(a) = f_2(a)$$

and denote this common value by μ. Let k be the integer for which

$$(k - 1/2)\pi \leqq \mu < (k + 1/2)\pi$$

and suppose that

(ii) $$f_1(b_1) = f_2(b_2) = (k + 1/2)\pi.$$

Let

(iii) $$f'_1(x) > 0 \text{ in } a < x \leqq b_1, \qquad f'_2(x) > 0 \text{ in } a < x \leqq b_2,$$

and let

(iv) $$f'_1(a) \geqq f'_2(a) \geqq 0.$$

Finally, let

(v) $$f'_1(x_1) \geqq f'_2(x_2)$$

for every x_1 in $[a, b_1]$, where $x_2 = x_2(x_1)$ denotes the uniquely determined point of $[a, b_2]$ for which

$$f_1(x_1) = f_2(x_2).$$

[This function $x_2(x_1)$ exists and is unique in $[a, b_2]$ by virtue of (iii)]. Then

(2) $$\left| \int_a^{b_1} \cos f_1(x)\,dx \right| \leqq \left| \int_a^{b_2} \cos f_2(x)\,dx \right|.$$

Proof. We notice first that these conditions cannot all be satisfied unless $b_2 \geqq b_1$. In fact, an application of the mean value theorem of the differential calculus to the inverse functions of $f_1(x)$ and $f_2(x)$ shows that the function $\phi(x_1)$ defined by

$$\phi(x_1) = x_2 - x_1 \equiv x_2(x_1) - x_1$$

is a monotone, non-decreasing function of x_1 in $[a, b_1]$ by virtue of condition (v). It is also clear from (iii)-(v) that

$$f_1(x) \geqq f_2(x) \text{ in } [a, b_1]. \tag{3}$$

In the sequel we shall suppose that $k = 0$. This is permissible, since

$$\left| \int_a^b \cos f(x)\,dx \right| = \left| \int_a^b \cos \{f(x) - k\pi\}\,dx \right|$$

for any integer k.

We now consider the cases $\mu \geqq 0$ and $\mu < 0$ separately.

If $\mu \geqq 0$, then (2) is trivial, for $\cos x$ is non-negative and decreasing in $[0, \pi/2]$, so that (3) implies

$$\cos f_1(x) \leqq \cos f_2(x) \text{ in } [a, b_1].$$

Consequently

$$\int_a^{b_2} \cos f_2(x)\,dx \geqq \int_a^{b_1} \cos f_2(x)\,dx \geqq \int_a^{b_1} \cos f_1(x)\,dx > 0.$$

If $\mu < 0$, then there exist, by virtue of (i)-(v), constants c_1, c_2, d, such that $a < c_1 < d < c_2 < b_2$, $d < b_1$, and

$$f_1(c_1) = f_2(c_2) = 0, \qquad f_1(d) = -f_2(d). \tag{4}$$

Since $\cos x$ is increasing in $[-\pi/2, 0]$ and decreasing in $[0, \pi/2]$, it is clear from (3) and (4) that

$$\cos f_1(x) \geqq \cos f_2(x) \text{ in } [a, d], \qquad \cos f_1(x) \leqq \cos f_2(x) \text{ in } [d, b_1]. \tag{5}$$

Furthermore, from (i) and (4),

$$\cos f_1(a) = \cos f_2(a), \qquad \cos f_1(d) = \cos f_2(d).$$

Hence, on denoting by A_1 (> 0) the area of the region bounded by the two curves

$$y = \cos f_1(x) \text{ in } [a, d], \qquad y = \cos f_2(x) \text{ in } [a, d],$$

and by A_2 (> 0) the area of the region bounded by the three curves

$y = \cos f_1(x)$ in $[d, b_1]$, $y = \cos f_2(x)$ in $[d, b_2]$, $y = 0$ in $[b_1, b_2]$,

we clearly have

$$\int_a^{b_2} \cos f_2(x)\,dx - \int_a^{b_1} \cos f_1(x)\,dx = A_2 - A_1.$$

Consequently, if we call A_3 (> 0) the area of the region bounded by the three curves

$y = \cos f_1(x)$ in $[c_1, d]$, $y = \cos f_2(x)$ in $[d, c_2]$, $y = 1$ in $[c_1, c_2]$,

then

$$(6) \qquad \int_a^{b_2} \cos f_2(x)\,dx - \int_a^{b_1} \cos f_1(x)\,dx = (A_2 + A_3) - (A_1 + A_3).$$

But $A_1 + A_3$ is precisely the area of the region bounded by the three curves

$y = \cos f_1(x)$ in $[a, c_1]$, $y = \cos f_2(x)$ in $[a, c_2]$, $y = 1$ in $[c_1, c_2]$,

and $A_2 + A_3$ is the area of the region bounded by the four curves

$y = \cos f_1(x)$ in $[c_1, b_1]$, $y = \cos f_2(x)$ in $[c_2, b_2]$, $y = 1$ in $[c_1, c_2]$, $y = 0$ in $[b_1, b_2]$.

Since $\phi(x_1)$ is a monotone, non-decreasing function on $[a, b_1]$, it follows that

$$A_1 + A_3 \leqq (c_2 - c_1)(1 - \cos \mu) \leqq c_2 - c_1$$

and

$$A_2 + A_3 \geqq c_2 - c_1.$$

Thus the expression on the right of (6) is non-negative, and (2) follows.

We now proceed with the proof of the theorem stated at the beginning. We may divide the interval $[a, b]$ into two parts corresponding to $f'(x) \leqq 0$ and $f'(x) \geqq 0$ and notice that it is clearly sufficient to prove the following theorem.

THEOREM. *If $f(x)$ is a real-valued, monotone, non-decreasing function possessing a second derivative $f''(x) \geqq r > 0$ in $[a, b]$, then*

$$\left| \int_a^b \cos f(x)\,dx \right| \leqq \int_0^{\beta_0} \cos\,(rx^2/2 + \mu_0)\,dx = \gamma_0/2r^{\frac{1}{2}}$$

where β_0, μ_0, γ_0 are the absolute constants defined above.

Proof. It is obviously no restriction to choose $a = 0$. Also, on placing $f(0) = \mu$, we may suppose, exactly as in the proof of the lemma, that

(7) $$-\pi/2 \leqq \mu < \pi/2.$$

Let

$$[0, b] = \sum_{j=0}^{m} [\delta_j, \delta_{j+1}]$$

be a subdivision of $[0, b]$ where the division points δ_j are determined by

(8) $$\delta_0 = 0; \quad \delta_{m+1} = b; \quad f(\delta_j) = (j - 1/2)\pi \qquad (j = 1, 2, \cdots, m)$$

so that if x is in $[\delta_j, \delta_{j+1}]$ then

$$(j - 1/2)\pi \leqq f(x) < (j + 1/2)\pi.$$

[If $f(b) \leqq \pi/2$, we put $\delta_1 = b$, and the inequality (11) reduces to $\left| \int_0^b \cos f(x)\,dx \right| \leqq \left| \int_0^{\delta_1} \cos f(x)\,dx \right|$, which is obvious.]

Since $f'(x)$ is non-negative and increasing throughout $[0, b]$, conditions (i)-(v) of the lemma are seen to be satisfied by the pair of functions

$$f_1(x) = f(x + \delta_{j+1} - \delta_j) - \pi; \quad f_2(x) = f(x)$$

in the pair of intervals

$$[a, b_1] = [\delta_j, \delta_{j+2} - \delta_{j+1} + \delta_j]; \quad [a, b_2] = [\delta_j, \delta_{j+1}],$$

for $(j = 1, 2, \cdots, m - 2)$. Hence

(9) $$\left| \int_{\delta_j}^{\delta_{j+1}} \cos f(x)\,dx \right| \geqq \left| \int_{\delta_{j+1}}^{\delta_{j+2}} \cos f(x)\,dx \right| \quad (j = 1, 2, \cdots, m - 2).$$

Although the lemma is not applicable in the case $j = m - 1$, it is quite obvious that (9) holds also for this value of j. In fact, we could extend the region of definition of $f(x)$ to $[0, \delta'_{m+1}]$ where

$$\delta'_{m+1} \geqq \delta_{m+1} = b, \qquad f(\delta'_{m+1}) = (m + 1/2)\pi;$$

and show

$$\left| \int_{\delta_m}^{b} \cos f(x)\,dx \right| \leqq \left| \int_{\delta_m}^{\delta'_{m+1}} \cos f(x)\,dx \right| \leqq \left| \int_{\delta_{m-1}}^{\delta_m} \cos f(x)\,dx \right|.$$

We now write

(10) $$\int_0^b \cos f(x)\,dx = \int_0^{\delta_1} \cos f(x)\,dx + \sum_{j=1}^{m} \int_{\delta_j}^{\delta_{j+1}} \cos f(x)\,dx$$

and notice that the expression on the right is an alternating sum, since

$\cos f(x)$ is of different sign in two successive subintervals $[\delta_j, \delta_{j+1}]$ and $[\delta_{j+1}, \delta_{j+2}]$. Further, if we exclude the first term of this expression, the remaining m terms are of decreasing absolute value by (9). Hence, on placing

$$u_1 = \int_0^{\delta_1} \cos f(x)\,dx > 0, \qquad u_2 = \int_{\delta_1}^{\delta_2} \cos f(x)\,dx < 0,$$

we obtain

$$u_1 \geqq \int_0^b \cos f(x)\,dx \geqq u_1 + u_2 > u_2$$

and consequently

$$(11) \quad \left| \int_0^b \cos f(x)\,dx \right| \leqq \max \left\{ \left| \int_0^{\delta_1} \cos f(x)\,dx \right| , \left| \int_{\delta_1}^{\delta_2} \cos f(x)\,dx \right| \right\}.$$

We now show that the conditions of the lemma are satisfied by the pair of functions

$$f_1(x) = f(x); \quad f_2(x) = rx^2/2 + \mu$$

in the pair of intervals

$$[a, b_1] = [0, \delta_1]; \; [a, b_2] = [0, \beta], \text{ where } \beta = [(\pi - 2\mu)r]^{\frac{1}{2}},$$

if one chooses the integer k occurring in the lemma as zero. In fact, (i) is obviously satisfied since $f_1(0) = \mu$ by the definition of μ. Condition (ii) is satisfied since $f_1(\delta_1) = \pi/2$ from (8). The assumption of monotony of $f(x)$ insures conditions (iii), (iv) of the lemma, since $r > 0$. In order to verify that (v) also is satisfied we write

$$f_1''(x) = r + g(x),$$

so that $g(x) \geqq 0$ in $[0, b]$ by assumption. This gives

$$f_1'(x) = rx + \int_0^x g(t)\,dt + f_1'(0),$$

where $f_1'(0) \geqq 0$ by the monotony of $f(x)$ in $[0, b]$. Also

$$f_1(x) = rx^2/2 + \int_0^x \int_0^y g(t)\,dt\,dy + f_1'(0)x + \mu.$$

Thus the definition $f_2(x_2) = f_1(x_1)$ of x_2 leads to

$$x_2 = \left\{ x_1^2 + \frac{2}{r} \int_0^{x_1} \int_0^y g(t)\,dt\,dy + \frac{2x_1}{r} f_1'(0) \right\}^{\frac{1}{2}},$$

and by substituting this into $f_2'(x) = rx$ we check that

$$f'_1(x_1) \geqq f'_2(x_2),$$

which is (v). The lemma then gives

$$\left| \int_0^{\delta_1} \cos f(x)\,dx \right| \leqq \int_0^{\beta} \cos\,(rx/2 + \mu)\,dx, \tag{12}$$

where

$$\beta = [(\pi - 2\mu)/r]^{1/2}.$$

It is similarly shown that

$$\left| \int_0^{\delta_2 - \delta_1} \cos\{f(x + \delta_1) - \pi\}\,dx \right| \leqq \int_0^{(2\pi/r)^{1/2}} \cos\left(\frac{rx^2 - \pi}{2}\right) dx. \tag{13}$$

The two inequalities (12) and (13) together with (11) give

$$\left| \int_0^{b} \cos f(x)\,dx \right| \leqq \max H(\mu)/2r^{1/2}, \qquad -\pi/2 \leqq \mu < \pi/2 \tag{14}$$

where $H(\mu)$ is independent of r, since

$$H(\mu) = 2\sqrt{r}\int_0^{\beta} \cos\,(rx^2/2 + \mu)\,dx, \qquad \beta = [(\pi - 2\mu)/r]^{1/2}.$$

It may be shown by direct calculation of $H'(\mu)$ that there exists one and only one maximum point for $H(\mu)$ on the range $[-\pi/2, \pi/2]$, say at the point $\mu = \mu_0$ where μ_0 satisfies the equation

$$\int_0^{(\pi/2 - \mu_0)^{1/2}} \sin\,(x^2 + \mu_0)\,dx = 0.$$

Then if we choose $\mu = \mu_0$, the inequality (14) must hold for all functions $f(x)$ satisfying the conditions of the theorem, and

$$H(\mu_0) = 2\sqrt{2}\int_0^{(\pi/2 - \mu_0)^{1/2}} \cos\,(x^2 + \mu_0)\,dx$$

is the desired least permissible value γ_0 of all numbers γ satisfying (1).

THE JOHNS HOPKINS UNIVERSITY.

SPECTRAL THEORY FOR A CERTAIN CLASS OF NON-SYMMETRIC COMPLETELY CONTINUOUS MATRICES.

By ANNA PELL WHEELER.

In this paper we consider the problem of the existence of characteristic numbers of a matrix A, which has real elements a_{ik} satisfying the condition

$$(\alpha) \qquad a_{ik} = a_{ki}, \quad i, k > 1; \qquad a_{1i} = -a_{i1}, \quad i > 1;$$

and which is completely continuous [1] with respect to the Hilbert space, i. e., the space of vectors $X = (x_1, x_2, \cdots)$ such that $\Sigma \mid x_i^2 \mid < \infty$. A vector of the Hilbert space shall be called an *H-vector*. As is to be expected, the characteristic numbers are not all necessarily real nor simple poles of the resolvent. The convergence of expansions in terms of characteristic vectors is established for certain iterated matrices of A.

1. *Character and existence of characteristic numbers.* We shall denote by AX the linear transformation $(\Sigma_k a_{ik} x_k)$ of the vector $X = (x_1, x_2, \cdots)$. So the system of linear equations

$$(1_1) \qquad \lambda x_1 = a_{11} x_1 - \sum_k a_{k1} x_k,$$

$$(1_2) \qquad \lambda x_i = a_{i1} x_1 + \sum_{k=2}^{\infty} a_{ik} x_k, \quad i > 1,$$

can be expressed by

$$(1) \qquad \lambda X = AX.$$

If the equations

$$(2) \qquad \begin{aligned} \lambda X^{(1)} &= AX^{(1)}, \\ X^{(1)} + \lambda X^{(2)} &= AX^{(2)}, \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ X^{(r)} + \lambda X^{(r+1)} &= AX^{(r+1)}, \end{aligned}$$

have solutions $X^{(1)}, X^{(2)}, \cdots X^{(r)}$ in the Hilbert space, but the $(r+1)$-th equation does not have a solution in the Hilbert space, the characteristic number λ shall be said to be *of rank* r with respect to X_1, or simply *of rank* r.

If the system (1) has a solution X for λ, the adjoint system

[1] Hellinger-Toeplitz, "Integralgleichungen und Gleichungen mit unendlichen Unbekannten," *Enzyklopädie der Mathematischen Wissenschaften*, Bd. II$_3$, Heft 9, p. 1400.

$$\lambda X^* = A'X^* \tag{3}$$

has a solution $X^* = (-x_1, x_2, \cdots)$. The case in which a characteristic number of A is of rank > 1 with respect to X can arise only when

$$(X, X^*) = -x_1^2 + \sum_{k=2}^{\infty} x_k^2 = 0. \tag{4}$$

There can be only one characteristic number of rank > 1. Assume that both λ and μ are of rank > 1 with respect to X and Y respectively. These vectors satisfy the conditions

$$(X, Y^*) = 0, \quad (X, X^*) = 0, \quad (Y, Y^*) = 0, \tag{5}$$

and from (5) it follows that

$$-(kx_1 + ly_1)^2 + \sum_{i=2}^{\infty} (kx_i + ly_i)^2 = 0 \tag{6}$$

for all k and l. If now k and l are so chosen that the first term of (6) vanishes, then all the other terms must vanish, and the vectors X and Y would not be linearly independent, and hence could not belong to different characteristic numbers. A characteristic number cannot be of rank > 1 with respect to two different vectors X and Y, for again the conditions (5) would hold.

A characteristic number cannot be of rank > 3. If the system (2) had solutions $X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}$, then $(X^{(1)}, X^{(1)*}) = 0$, $(X^{(2)}, X^{(1)*}) = 0$, and $(X^{(3)}, X^{(1)*}) = 0$. From the third equation of (2),

$$\begin{aligned}(X^{(2)*}, X^{(2)}) + \lambda(X^{(2)*}, X^{(3)}) &= (X^{(2)*}, AX^{(3)}) = (X^{(3)}, A'X^{(2)*}) \\ &= (X^{(3)}, X^{(1)*}) + \lambda(X^{(3)}, X^{(2)*}),\end{aligned}$$

and therefore $(X^{(2)}, X^{(2)*}) = 0$. It follows as in the two previous cases that $X^{(1)}$ and $X^{(2)}$ cannot be linearly independent. That a characteristic number may be of rank three is easily shown by special cases. There can be at most one pair of conjugate imaginary characteristic numbers. For, let $\lambda, \bar{\lambda}$ and $\mu, \bar{\mu}$ be two pairs of conjugate imaginary characteristic numbers, and let $X, \bar{X}$ and $Y, \bar{Y}$ be the corresponding characteristic vectors. Then $(X, \bar{X}^*) = 0$, $(Y, \bar{Y}^*) = 0$, $(X, \bar{Y}^*) = 0$, and if the constants $k = k_1 + ik_2$, $l = l_1 + il_2$ are chosen so that $kx_1 + ly_1 = 0$, the two vectors X and Y cannot be linearly independent. We have the following theorem.

THEOREM 1. *If the matrix A satisfies the condition (α) and is a completely continuous matrix with respect to the Hilbert space, there can be no more than one characteristic number of rank greater than unity, no characteristic number of rank greater than three, no characteristic number of rank greater than one with respect to more than one vector, and no more than one pair of conjugate imaginary characteristic numbers.*

For $i, k > 1$ the matrix (a_{ik}) is symmetric and completely continuous with respect to the Hilbert space and hence has characteristic [2] numbers μ_α, which are all real, and $\Sigma_\alpha \mu_\alpha^2 < \infty$. Let L_α be the corresponding characteristic vectors, and we may assume that they form an orthonormal system, $(L_\alpha, L_\beta) = \delta_{\alpha\beta}$. The solution of (1_2), regarded as a non-homogeneous system in $x_i, i > 1$, is given by

$$\lambda x_i = x_1 a_{i1} + x_1 \sum_\alpha \frac{\mu_\alpha l_{\alpha i} \sum_2 l_{\alpha k} a_{k1}}{\lambda - \mu_\alpha}, \quad i > 1. \tag{7}$$

The substitution of (7) in (1) gives for $x_1 \neq 0$,

$$D(\lambda) = \lambda(\lambda - a_{11}) + \sum_2^\infty a^2{}_{1i} + \sum_\alpha \frac{\mu_\alpha \left(\sum_2 l_{\alpha i} a_{1i} \right)^2}{\lambda - \mu_\alpha} = 0, \tag{8}$$

a necessary and sufficient condition which λ must satisfy to be a characteristic number of A. The function $D(\lambda)$ is continuous in the intervals

$$\mu_{\alpha-1} + \epsilon \leqq \lambda \leqq \mu_\alpha - \epsilon, \quad \epsilon > 0$$

and changes sign between $\mu_{\alpha-1}$ and μ_α. Therefore there exists at least one characteristic number for A between two consecutive positive or two consecutive negative numbers μ_α. Hence the matrix A has real characteristic numbers except perhaps when the matrix (a_{ik}), $i, k > 1$, has only one or no characteristic numbers, or only one positive and one negative. In these three exceptional cases the equation (8) reduces to an algebraic equation. It can easily be shown that if the only root of (8) is zero, it is of rank greater than unity. In the sequel zero will be referred to as a characteristic number only if it is of rank greater than unity. Hence

THEOREM 2. *If a matrix A is not the zero matrix, satisfies the conditions (α) and is completely continuous with respect to the Hilbert space, then it has at least one characteristic number.*

2. *Form of the expansion.* If the vectors T_p and T_q belong to different characteristic numbers,

$$(T_p, T_q^*) = 0. \tag{9}$$

Whenever two vectors satisfy (9), and $(T_p, T_p^*) = 0$ or a negative value, then (T_q, T_q^*) must be positive, for otherwise constants c_1 and c_2 not both zero could be chosen so that $\Sigma_i (c_1 t_{pi} + c_2 t_{qi})^2$ would be zero or negative. Hence if T_p and T_q belong to a characteristic number of rank unity, it may be assumed that

[2] Hellinger-Toeplitz, *loc. cit.*, p. 1553.

$$(T_p, T_q^*) = \pm \delta_{pq}, \tag{10}$$

where the minus sign holds for only one vector. Also if λ is of rank > 1, linear combinations T_p of the solutions of (2) can be formed so that they satisfy (10).

If λ_a is a characteristic number of rank unity, let

$$E_p(X, X') = \lambda_p \sum_{a=1}^{s} \pm (T_p^{(a)}, X)(T_p^{(a)*}, X'), \tag{11}$$

where the $T_p^{(a)}$ form a fundamental system of characteristic vectors for λ_a, and the minus sign in (11) occurs before a term only if in that term the T_p satisfies $(T_p, T_p^*) = -1$.

If λ_i and $\bar{\lambda}_i$ form a pair of conjugate imaginary characteristic numbers, let

$$E_i(X, X') = \pm [\lambda_i (T^{(i)}, X)(T^{(i)*}, X') + \bar{\lambda}_i (\bar{T}^{(i)}, X)(\bar{T}^{(i)*}, X')], \tag{12}$$

where $$(T^{(i)}, \bar{T}^{(i)*}) = 0 \text{ and } (T^{(i)}, T^{(i)*}) = \pm 1.$$

If λ_m is a characteristic number of rank $r > 1$, let

$$E_m(X, X') = \sum_{p,q=1}^{r} \pm c_{pq}(T_p^{(m)}, X)(T_q^{(m)*}, X'), \tag{13}$$

where $T_p^{(m)}$ are linear combinations of the solutions of (2) which satisfy (10); $AT_p^{(m)} = \Sigma_q c_{pq} T_q^{(m)}$; $c_{pq} = c_{qp}$; and the determinant $|c_{pq}| \neq 0$.

If $\lambda = 0$ is a characteristic number of rank $r > 1$, let

$$E_0(X, X') = \sum_{p,q=1}^{r} \pm c_{pq}(T_p^{(0)}, X)(T_q^{(0)*}, X'), \tag{14}$$

where the $T_p^{(0)}$ satisfy (10); $AT_p^{(0)} = \Sigma_q c_{pq} T_q^{(0)}$; $c_{pq} = c_{qp}$; and the determinant $|c_{pq}| = 0$.

The matrices E_p, E_i, E_m, E_0 are mutually orthogonal. In considering the expansion of $A(X, X')$ the question of convergence enters only if there is an infinite number of characteristic numbers of rank $= 1$, for then there would be an infinite series of the form

$$\sum_p E_p(X, X'). \tag{15}$$

We shall now prove the following theorem.

THEOREM 3. *If the matrix A satisfies condition (α), is completely continuous with respect to the Hilbert space, and if the series* (15) *converges to a completely continuous bilinear form, then*

$$A(X, X') = \sum_p E_p(X, X') + E_i(X, X') + E_m(X, X') + E_0(X, X') \tag{16}$$

for any H-vectors X and X'.

Let $B(X, X')$ be the bilinear form formed by the difference of the two sides of (16). By hypothesis B is completely continuous, and the elements b_{ik} satisfy condition (α). Hence by Theorem 2 if B is not the zero matrix, there exists at least one characteristic number. Let F_p, F_i, F_m, F_0 U_p bear the same relation to B as E_p, E_i, E_m, E_0, T_p do to A. The vectors T_p are orthogonal to B and therefore to all the vectors U_p, and hence to the F's. On the other hand the U_p satisfy $AU_p = BU_p$ so that the U_p are linearly dependent on the T_p, and there is a contradiction. It follows that B is the zero matrix and the theorem is proved.

3. *Convergence of the expansion.* The iterated matrix $A^{(2)} = (\Sigma_j a_{ij} a_{jk})$ is completely continuous with respect to the Hilbert space, and satisfies the condition (α). If $\lambda \neq 0$ is a characteristic number of rank r for A, then λ^2 is a characteristic number of rank r for $A^{(2)}$, and if λ is a characteristic number for $A^{(2)}$, with U_p as a characteristic vector, then $+\lambda^{\frac{1}{2}}$ or $-\lambda^{\frac{1}{2}}$, or both are characteristic numbers of A, with u or $u^{\frac{1}{2}} + Au$ as characteristic vectors. If $\lambda = 0$ is a characteristic vector of A of rank r, then $\lambda = 0$ is a characteristic number of rank $r - 1$ of $A^{(2)}$. The matrix $A^{(2)}$ is not the zero matrix unless $\lambda = 0$ is the only characteristic number of A and is of rank $\leqq 2$. Similar results hold for the iterated matrices $A^{(n)}$; in particular zero is not a characteristic number of $A^{(n)}$ if $n > 3$.

THEOREM 4. *If the matrix A satisfies the condition (α), and is completely continuous with respect to the Hilbert space, and if the vector*

$$X^{(0)} = (1, \sum_{i=2}^{\infty} \sum_p (a_{1i} l_{pi} l_{pk}/\mu_p)) \tag{17}$$

does not satisfy

$$(X^{(0)}, X^{(0)*}) = 0, \tag{18}$$

and

$$AX^{(0)} = 0, \tag{19}$$

then the right hand side of

$$A^{(2)}(X, X') = \sum_p E_p^{(2)}(X, X') + E_i^{(2)}(X, X') + E_m^{(2)}(X, X') + E_0^{(2)}(X, X') \tag{20}$$

converges absolutely, and to the left hand side for any two H-vectors X and X'. If the vector $X^{(0)}$ does satisfy (18) *and* (19), *then the right hand side of*

$$(21) \qquad A^{(4)}(X, X') = \sum_p E_p^{(4)}(X, X') + E_i^{(4)}(X, X') + E_m^{(4)}(X, X')$$

converges absolutely, and to the left hand side for any two H-vectors X and X'.

To prove the absolute convergence of (20), it is sufficient to show that the matrix [3] $(\lambda_p t_{pi})$ is completely continuous. Only those terms in E_p for which

$$(22) \qquad (T_p, T_p^*) = \sum_{i=2}^{\infty} t_{pi}^2 - t_{p1}^2 = 1$$

need be considered. And from (22) it follows that the matrix $(\lambda_p t_{pi})$ is completely continuous if t_{p1} is bounded, for then $\Sigma_{ip}\lambda_p^2 t_{pi}^2$ converges. Assume first that the λ_p are all distinct from μ_a. The components of the vectors T_p are given by

$$(23) \qquad t_{pi} = -\frac{t_{p1}}{\lambda_p}\left[a_{1i} + \sum_q \frac{\mu_q l_{qi} c_q}{\lambda_p - \mu_q}\right], \quad i > 1,$$

where $c_p = \Sigma_2 l_{pi} a_{1i}$, and the substitution of (23) in (22) gives

$$(24) \qquad t_{p1}^2\left[-1 + \frac{\sum_2^{\infty} a_{1i}^2 - \sum_1^{\infty} c_q^2}{\lambda_p^2} + \sum_q \frac{c_q^2}{(\lambda_p - \mu_q)^2}\right] = 1.$$

The sequence t_{p1} is bounded unless

$$(25) \qquad \sum_i a_{1i}^2 = \sum_p c_p^2,$$

and

$$(26) \qquad \lim_{p' \to \infty} \sum_q \frac{c_q^2}{(\lambda_{p'} - \mu_q)^2} = 1,$$

for some subsequence $t_{p'1}$. We wish to show that under the conditions (25) and (26), the vector $X^{(0)}$ of (17) satisfies (18) and (19). The condition (25) says that the vector $X^{(0)}$ satisfies all except possibly the first equation of (19). From (26) we obtain

$$(27) \qquad \sum_q \frac{c_q^2}{\mu_q^2} \leqq 1,$$

and hence the absolute convergence of

$$(28) \qquad \sum_q \frac{c_q^2}{\mu_q} \leqq \sum_q c_q^2.$$

The equation (8) can now be written

$$(8') \qquad D(\lambda_{p'}) = -a_{11} - \lambda_{p'} + \sum_q \frac{c_q^2}{\mu_q} + \lambda_{p'} \sum_q \frac{c_q^2}{(\lambda_{p'} - \mu_q)\mu_q} = 0.$$

[3] For convenience the superscript α is omitted from $t_{pi}^{(\alpha)}$.

On account of (26) and (27) the coefficient of $\lambda_{p'}$ in the last term of (8′) is bounded and therefore

$$a_{11} = \sum_q \frac{c_q^2}{\mu_q}, \tag{29}$$

which expresses the fact that $X^{(0)}$ satisfies the first equation of (19). In virtue of (29) the equation (8′) takes the form

$$\lambda_{p'}\left(1 - \sum_q \frac{c_q^2}{(\lambda_{p'} - \mu_q)\mu_q}\right) = 0 \tag{8''}$$

and this combined with (26) and (27) gives

$$\sum_q \frac{c_q^2}{\mu_q^2} = 1, \tag{30}$$

which says that the vector $X^{(0)}$ satisfies (18). This completes the proof of the first part of the theorem.

For the second part of the theorem, it is sufficient to prove the complete continuity of the matrix $(\lambda_{p'}^2 t_{p'i})$ under the conditions (25) and (26). From (8″) and (30)

$$\sum_q \frac{c_q^2}{(\lambda_{p'} - \mu_q)\mu_q^2} = 0, \tag{31}$$

and hence

$$\begin{aligned} \sum_q \frac{c_q^2}{(\lambda_{p'} - \mu_q)^2} - 1 &= \sum \frac{c_q^2}{(\lambda_{p'} - \mu_q)^2} - \sum \frac{c_q^2}{\mu_q^2} \\ &= \lambda_{p'}^2 \sum \frac{c_q^2}{\mu_q^2(\lambda_{p'} - \mu_q)^2} \geqq \frac{\lambda_{p'}^2}{\sum c_q^2} \end{aligned} \tag{32}$$

on account of (31) and (8′). And since $t^2_{p'1}$ is the reciprocal of (32) under (25) and (26), it follows that

$$\lambda_{p'}^2 t_{p'_1}^2 \leqq \sum_q c_q^2,$$

This shows that the matrix $(\lambda_{p'}^2 t_{p'i})$ is completely continuous, and the second part of the theorem is now a consequence of Theorem 3.

It remains to prove the theorem for the case in which an infinite number of λ's are equal to μ_q. Let the other characteristic numbers of the matrix (a_{ik}), $i, k > 1$, be denoted by μ_p. Since $c_q = \Sigma_2 l_{qi} a_{i1} = 0$, the system (1) has the solutions $V_p = (0, l_{p1}, l_{p2}, \cdots)$ for which the matrix v_{qi} is bounded and $\lambda_q v_{qi}$ is completely continuous. If other solutions T_q exist for $\lambda = \mu_q$, the complete continuity of $\lambda_q t_{qi}$ and $\lambda_q^2 t_{qi}$ can be shown in the same way as above.

BRYN MAWR COLLEGE.

ON SUMMATION OF DERIVED SERIES OF THE CONJUGATE FOURIER SERIES.*

By A. F. Moursund.

1. *Introduction.* This paper is a continuation of an earlier paper in which we give theorems concerning the summability of the r-th, $r \geqq 0$, derived series of the conjugate series of the Fourier series generated by a Lebesgue integrable function.[1] Our principal results are three theorems concerning the effectiveness of the N_{z_p} summation method[2] for the summation of the r-th derived series of the conjugate Fourier series. These theorems and the principal theorem of our earlier paper[3] contain, as special cases, results, some of which are new, for the Bosanquet-Linfoot and Cesàro summation methods.[4] The case $r = 0$ gives us three well known and one new theorem concerning the Cesàro summability of the conjugate Fourier series.

2. *Notation.* Throughout this paper we consider $f(x)$ to be Lebesgue integrable on $(-\pi, \pi)$ and periodic of period 2π. We use, as far as possible, the notation of I;[5] and in addition set:

$$B_r(s) \equiv \int_0^s |A_r(s)|/s^r \, ds,$$

$$A^*_r(s) \equiv \int_0^s A_r(s)\, ds,$$

$$B^*_r(s) \equiv \int_0^s |A^*_r(s)|/s^{r+1} \, ds;$$

$$C^*_r \equiv \begin{cases} C_r & (r \text{ even}) \\ C_r - A^*_r(\pi) \dfrac{d^r}{ds^r} \cot s/2 \Big]_{s=\pi} & (r \text{ odd}); \end{cases} \tag{2.2}$$

(2.3)

* Presented to the American Mathematical Society, October 27, 1934.

[1] A. F. Moursund, "On summation of derived series of the conjugate Fourier series," *Annals of Mathematics* (2), vol. 36 (1935), pp. 182-193. Throughout this paper the paper cited here will be referred to as I.

[2] A. F. Moursund, "On the Nevanlinna and Bosanquet-Linfoot summation methods," *Annals of Mathematics* (2), vol. 35 (1934), pp. 239-247 (Section II).

[3] *Loc. cit.* [1], Theorem 10. 1.

[4] The question of inclusion of other methods by the N_{z_p} method is discussed in *loc. cit.* [2], Section II.

[5] See § 2. To avoid confusion, the functions there designated by $A^*_r(s)$ and $B_r(n)$ should be designated by other letters, and the functions $F_i(ns)$ and $G_i(ns)$ should be designated, respectively, as $F(i, r+1, ns)$ and $G(i, r+1, ns)$.

$$(2.3)\qquad D^*_r(n) \equiv \begin{cases} D_r(n) & (r \text{ even}) \\ D_r(n) - A^*_r(\pi)\Big[\sum_{j=0}^{(r-1)/2} \dfrac{r!}{(2j+1)!\,(r-2j-1)!}\,\dfrac{d^{2j+1}}{ds^{2j+1}}\cot s/2 \\ \qquad \times \dfrac{d^{r-2j-1}}{ds^{r-2j-1}}\cos ns + \dfrac{d^r}{ds^r}\sin ns\Big]_{s=\pi} & (r \text{ odd}); \end{cases}$$

$$(2.4)\qquad \tilde{f}^{*(r)}(x) \equiv -\lim_{\epsilon\to 0}\int_\epsilon^\pi A^*_r(s)\,\frac{d^{r+1}}{ds^{r+1}}\cot s/2\,ds - C^*_r;$$

and

$$(2.5)\qquad F(i,p,v) \equiv \int_0^1 z_p(t)\,t^i\,\frac{d^i}{dt^i}\cos vt\,dt,$$

$$G(i,p,v) \equiv \int_0^1 z_p(t)\,t^i\,\frac{d^i}{dt^i}\sin vt\,dt.$$

3. *Expressions for* $\sigma_n^{(r)}(x)$ *and* $N_{z_r}\sigma_n^{(r)}(x)$. In § 3 of I we showed that

$$(3.1)\qquad \sigma_n^{(r)}(x) = \int_0^\pi A_r(s)\,\frac{d^r}{ds^r}\,\theta(s,n)\,ds - C_r + D_r(n).$$

Proceeding as we did there in deriving (3.6), we have

$$\begin{aligned}(3.2)\qquad N_{z_r}\sigma_n^{(r)}(x) &= \int_{1/n^2}^\pi A_r(s)\,\frac{d^r}{ds^r}\cot s/2\,ds - C_r \\ &+ \int_0^1 z_r(t)\,dt\int_0^{1/n^2} A_r(s)\,\frac{d^r}{ds^r}\,\theta(s,nt)\,ds \\ &- \sum_{i=0}^{r-1}\frac{r!}{i!\,(r-i)!}\Big[\int_{1/n^2}^\delta + \int_\delta^\pi\Big] A_r(s)/s^i\,\frac{d^{r-i}}{ds^{r-i}}\cot s/2\,F(i,r,ns)\,ds \\ &- \Big[\int_{1/n^2}^{A/n} + \int_{A/n}^\delta + \int_\delta^\pi\Big] A_r(s)/s^r\cot s/2\,F(r,r,ns)\,ds \\ &+ \Big[\int_{1/n^2}^{A/n} + \int_{A/n}^\delta + \int_\delta^\pi\Big] A_r(s)/s^r\,G(r,r,ns)\,ds \\ &+ \int_0^1 z_r(t)\,D_r(nt)\,dt \\ &\equiv \int_{1/n^2}^\pi A_r(s)\,\frac{d^r}{ds^r}\cot s/2\,ds - C_r + \sum_{i=1}^{10} J_i.\end{aligned}$$

Upon integrating (3.1) by parts we obtain, with the aid of Lemmas 3.1 and 3.2 of I,

$$(3.3)\qquad \sigma_n^{(r)}(x) = -\int_0^\pi A^*_r(s)\,\frac{d^{r+1}}{ds^{r+1}}\,\theta(s,n)\,ds - C^*_r + D^*_r(n).$$

Our use of (3.3) is analogous to our use of (3.1).

4. *Lemmas.* In proving our theorems we use, in addition to the lemmas of I, the following lemmas.[6]

LEMMA 4.1. *For v on $(0, \infty)$ when $0 \leqq i < p$ and on $(0, A)$ when $i = p$*

$$F(i, p, v) = P_{i_1}(v) - P_{i_2}(v)$$
$$G(i, p, v) = Q_{i_1}(v) - Q_{i_2}(v)$$

where the P's and Q's are bounded monotone decreasing positive functions of v.

LEMMA 4.2. *For A sufficiently large and $v \geqq A$,*

$$|F(p, p, v)|, \ |G(p, p, v)| \leqq K/v + Z_p(v)$$

where K is a constant.

Proof. For v sufficiently large

$$|F(p, p, v)| = \left| \int_0^1 z_p(t) t^p \frac{d^p}{dt^p} \cos vt \, dt \right|$$
$$= \left| \int_0^1 \frac{d^p}{dt^p} [z_p(t) t^p] \cos vt \, dt \right| \leqq K/v + Z_p(v).$$

The proof is similar for $G(p, p, v)$.

LEMMA 4.3. *$F(p, p, v)$ and $G(p, p, v) \to 0$ as $v \to \infty$.*

The functions which we consider in the remaining lemmas of this section are defined by (3.2).

LEMMA 4.4. *Wherever $\tilde{f}^{(r)}(x)$ exists J_4 and $J_7 \to 0$ as $n \to \infty$ for a fixed A.*

LEMMA 4.5. *At a point where $A_r(s) = 0(s^r)$, as $s \to 0$, J_5 and $J_8 \to 0$ as A, $n \to \infty$ for a fixed sufficiently small δ.*

Proof. Using Lemma 4.2 we have, for a fixed sufficiently small δ,

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[6] In proving the lemmas we use properties of the N_{z_p} method, *loc. cit.* [2], Section II, without further referring to them. The proof of Lemma 4.1 is like the proof of Lemma 6.3 and (6.1) of I; that of Lemma 4.4 like that of Lemma 8.2 of I. In the proof of Lemma 4.5 the last term $\to 0$ as A, $n \to \infty$ because $\int_C^\infty Z_r(s)/sr \, ds$ exists.

$$|J_5| \leqq 2\int_{A/n}^{\delta} |A_r(s)|/s^r[K/ns^2 + Z_r(ns)/s]ds$$
$$= 0\left[1/n\int_{A/n}^{\delta} 1/s^2\,ds + \int_{A/n}^{\delta} Z_r(ns)/s\,ds\right]$$
$$= 0\left[1/A + \int_{A}^{n\delta} Z_r(s)/s\,ds\right] = o(1) \text{ as } A,\ n\to\infty.$$

Similarly $J_6 \to 0$.

LEMMA 4.6. *At a point where $B_r(s) = o(s)$, as $s\to 0$, J_5 and $J_8 \to 0$ as $\delta\to 0$ and $n\to\infty$ for a fixed sufficiently large A.*

Proof. Integrating by parts we have

$$|J_5| \leqq 2\int_{A/n}^{\delta} |A_r(s)|/s^r[K/ns^2 + Z_r(ns)/s]ds$$
$$= 2B_r(s)/s[K/ns + Z_r(ns)]\Big]_{A/n}^{\delta} + 2K/n\int_{A/n}^{\delta} B_r(s)/s\cdot 1/s^2\,ds$$
$$-2\left[\int_{A/n}^{\delta} B_r(s)/s\,Z'_r(ns)\,ds + \int_{A/n}^{\delta} B_r(s)/s\,Z_r(ns)/s\,ds\right]$$
$$= o(1) + o(1) + o(1)Z_r(ns)\Big]_{A/n}^{\delta} + o(1)\int_{A/n}^{\delta} Z_r(ns)/s\,ds$$
$$= o(1) \text{ as } \delta\to 0 \text{ and } n\to\infty. \text{ Similarly } J_8 = o(1).$$

LEMMA 4.7. *J_6 and $J_9 \to 0$ as $n\to\infty$ for a fixed δ.*

Proof. The lemma is a consequence of Lemma 4.3.

5. *Theorems concerning the existence of $\tilde{f}^{*(r)}(x)$ and the behavior of the functions defined by* (2.1). In § 9 of I we gave sufficient conditions for the existence of $\tilde{f}^{(r)}(x)$.[7] We consider here some further consequences of these conditions.

THEOREM 5.1. *If $f(x)$ is such that $d^{r-1}f(x)/dx^{r-1}$ is of bounded variation on $(-\pi,\pi)$ then, as $s\to 0$, $A_r(s) = 0(s^r)$ and $B_r(s) = o(s)$ almost everywhere.*

Proof. By Lemma 7.6 of I $A_r(s) = o(s^r)$ when $d^rf(x)/dx^r$ exists. The theorem follows for when $d^{r-1}f(x)/ds^{r-1}$ is of bounded variation $d^rf(x)/dx^r$ exists almost everywhere.

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[7] In Lemma 7.8 and Theorem 9.3 of I the integral $\int_{\epsilon}^{\pi} dr/ds^r\Phi_r(s)\cot s/2\,ds$ should be replaced by $\int_{\epsilon}^{\pi} \cot s/2\,d[d^{r-1}/ds^{r-1}\Phi_r(s)]$; the existence almost everywhere of the limit as $\epsilon\to 0$ depends on Plessner's proof of the existence of $\tilde{f}'(x)$ rather than on his proof of the existence of $\tilde{f}(x)$.

THEOREM 5.2. *Wherever the generalized derivative* $f^{(r+1)}(x)$ *exists* $A_r(s) = 0(s^r)$ *and* $B_r(s) = o(s)$, *as* $s \to 0$.

Proof. Wherever $f^{(r+1)}(x)$ exists we see using Definition (2.1) of I that

$$A_r(s) = (-1)^{r+1}/\pi[f^{(r+1)}(x) + w(x,s)]s^{r+1}$$

where $w(x,s) \to 0$ as $s \to 0$. The theorem follows.[8]

THEOREM 5.3. *The existence of* $\bar{f}^{(r)}(x)$ *implies the existence of* $\bar{f}^{*(r)}(x)$, but the converse is not necessarily true.[9]

THEOREM 5.4. $A^*_r(s) = 0(s^{r+1})$ *wherever* $A_r(s) = 0(s^r)$ *and* $B^*_r(s) = o(s)$ *wherever* $B_r(s) = o(s)$, *but the converse is not necessarily true.*

Proof. Wherever $B_r(s) = o(s)$

$$B^*_r(s) \leqq \int_0^s 1/s^{r+1}\, ds \int_0^s |A_r(t)|\, dt \leqq \int_0^s B_r(s)/s\, ds = o(s).$$

The rest of the proof is left to the reader.

6. *Summability theorems.* In this section we give our principal theorems and, for convenience in reference, state also the principal theorem of I. We assume, as everywhere in this paper, that the function $f(x)$ which generates the series under consideration is Lebesgue integrable on $(-\pi, \pi)$ and of period 2π.

THEOREM 6.1. *The* N_{z_p} *sum of the r-th* $(r = 0, 1, 2, \cdots, p)$, *derived series of the conjugate Fourier series is* $\bar{f}^{(r)}(x)$ *wherever that limit exists and, as* $s \to 0$, *either* $A_r(s) = 0(s^r)$ *or* $B_r(s) = o(s)$.

THEOREM 6.2. *The* N_{z_p} *sum of the r-th* $(r = 0, 1, 2, \cdots, p-1)$, *derived series of the conjugate Fourier series is* $\bar{f}^{(r)}(x)$ *wherever that limit exists.*

Proof of Theorems 6.1 *and* 6.2. Theorem 6.2 is Theorem 10.1 of I. Referring to the proof of that theorem and (3.2) we see that at a point where $\bar{f}^{(r)}(x)$ exists J_1, J_2, J_3, and $J_{10} \to 0$, respectively, like K_1, K_2, K_3, and $K_6 \to 0$. When, also, either $A_r(s) = 0(s^r)$ or $B_r(s) = o(s)$ it follows from

[8] In fact $A_r(s) = 0(s^{r+1})$ and $B_r(s) = 0(s^2)$.

[9] See B. N. Prasad, "Contribution à l'étude de la Série Conjuguée d'une Série de Fourier," *Journal de Mathématiques Pures and Appliquées* (9), vol. 11 (1932), pp. 153-205 (p. 178).

Lemmas 4. 4, 4. 5, 4. 6, and 4. 7 that J_4 to J_9, inclusive, $\to 0$. Whence, as do corresponding functions in the proof of Theorem 10. 1,

$$N_{z_r}\sigma_n^{(r)}(x) \text{ and } N_{z_p}\sigma_n^{(r)}(x),\ p > r, \to \bar{f}^{(r)}(x) \text{ as } n \to \infty.$$

This completes the proof of Theorem 6. 1.

THEOREM 6. 3. *The N_{z_p} sum of the r-th $(r = 0, 1, 2, \cdots, p-1)$, derived series of the conjugate Fourier series is $\bar{f}^{*(r)}(x)$ wherever that limit exists and, as $s \to 0$, either $A^*_r(s) = 0(s^{r+1})$ or $B^*_r(s) = o(s)$.*

THEOREM 6. 4. *The N_{z_p} sum of the r-th $(r = 0, 1, 2, \cdots, p-2)$, derived series of the conjugate Fourier series is $\bar{f}^{*(r)}(x)$ wherever that limit exists.*

Proof of Theorems 6. 3 and 6. 4 are similar to those of Theorems 6. 1 and 6. 2. We first form from (3. 3) expressions for the $N_{z_{r+1}}$ and $N_{z_{r+2}}$ transforms of $\sigma_n^{(r)}(x)$ which are similar, respectively, to (3. 2) of this paper and (3. 6) of I; next prove lemmas similar to the lemmas of § 5 of I, replacing $A_r(s)$ by $A^*_r(s)$, $\frac{d^r}{ds^r}\cot s/2$ by $\frac{d^{r+1}}{ds^{r+1}}\cot s/2$, et cetera; and then proceed as in the proofs of Theorems 6. 1 and 6. 2.[10]

7. *Theorems for Bosanquet-Linfoot and Cesàro summability.* By choosing the functions

$$B_{\alpha,\beta}(t) = G(1-t)^{\alpha-1}(\log C/1-t)^{-\beta} \tag{7. 1}$$

with $\alpha > p$ or $\alpha = p$, $\beta > 1$, and $G = G(\alpha, \beta, C)$ such that

$$\int_0^1 B_{\alpha,\beta}(t)\,dt = 1;$$

and

$$(p+\delta)(1-t)^{p+\delta-1} \tag{7. 2}$$

with $\delta > 0$, respectively, as kernels for the N_{z_p} method we obtain theorems for the Bosanquet-Linfoot and Cesàro methods from the theorems of § 6.[11]

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[10] As a consequence of the Riemann-Lebesgue theorem Lemma 6. 5 of I remains valid when $z_{r+1}(t)$ is replaced by $z_r(t)$. Hence terms arising from $D^*_r(n) \to 0$ like those arising from $D_r(n)$.

[11] See *loc. cit.* [2], Section II. The Cesàro method theorems are obtained as a result of the equivalence of the Riesz and Cesàro methods.

Among the theorems thus obtained are three well known theorems concerning the Cesàro summability of the conjugate Fourier series[12] and the following new theorem.

THEOREM 7.1. *The conjugate Fourier series is summable* $(C, 2+\delta)$, *for every* $\delta > 0$, *to*

$$\tilde{f}^*(x) = \lim_{\epsilon \to 0} -1/4\pi \int_\epsilon^\pi \left\{ \int_0^s [f(x+t) - f(x-t)]\,dt \right\} \csc^2 s/2\, ds$$

wherever that limit exists.

Theorem 7.1 is related to the third of the Cesàro method theorems mentioned above (Prasad's theorem) like the second is to the first.

THE UNIVERSITY OF OREGON,
EUGENE, OREGON.

[12] See E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. II (2nd Edition), p. 697, for the first of these theorems which is the standard (C, δ), $\delta > 0$, summability theorem for the conjugate Fourier series; for the second see R. E. A. C. Paley, "On the Cesàro summability of Fourier series and allied series," *Proceedings of the Cambridge Philosophical Society*, vol. 26 (1930), pp. 173-203 (Theorem 2 with $\alpha = 0$); and for the third see Prasad, *loc. cit.* [9], Theorem IV.

A CERTAIN MIXED LINEAR INTEGRAL EQUATION.

By Olive Margaret Hughes.

Introduction. The purpose of this paper is the development of a spectral theory for the linear functional transformation

$$Tf(x) = MK(x, \xi)f(\xi) + \int_0^1 K(x, s)f(s)\,ds \tag{1}$$

of real-valued functions $f(x)$, continuous on the interval $I(0 \leq x \leq 1)$, where

(a) The kernel $K(x, s)$ is real, symmetric [1] and continuous on the square $R(0 \leq x \leq 1,\ 0 \leq s \leq 1)$.

(b) ξ is a given fixed value of the variable of integration in I.

(c) M is a known negative constant.

W. A. Hurwitz [2] has obtained some alternative theorems for such transformations. For M positive or zero the above problem reduces respectively to a Kneser weighted problem [3] and to the well-known Fredholm problem; more recent works [4] include both of these as special cases of a general spectral theory for linear symmetric functional transformations. They do not cover the case however with which we are concerned, since under the given hypothesis characteristic numbers of the integral equation

$$\lambda u(x) = MK(x, \xi)u(\xi) + \int K(x, s)u(s)\,ds \tag{2}$$

may be imaginary or zero, and may be poles of the resolvent of order greater than one. The following simple examples illustrate these possibilities:

[1] A definition of symmetric functions is given on page 871.

[2] W. A. Hurwitz, "Mixed linear integral equations of the first order," *Transactions of the American Mathematical Society*, vol. 16 (1915).

[3] A. Kneser, *Die Integralgleichungen*, Rendiconti, vol. 14 (1914). Also his *Die Integralgleichungen und ihre Anwendungen in der Mathematischen Physik*, 2 umgearbeitete Auflage (1922).

[4] J. von Neumann, "Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren," *Mathematische Annalen*, vol. 102. Also M. H. Stone, "Linear transformations in Hilbert space and their applications to analysis," *Colloquium Lectures of the American Mathematical Society*, vol. 15 (1932).

Ex. 1. If $K(x,s) = xs$, $M = -1/3$, $\xi = 1$, then $u(x) = x$ is a characteristic function corresponding to the characteristic number $\lambda = 0$ of the first order.

Ex. 2. If $K(x,s) = 1 - 4xs$, $M = -1/3$, $\xi = 1$, then $u(x) = 1 - 2x$ is a characteristic function corresponding to $\lambda = 1/3$ of order two.

Ex. 3. If $K(x,s) = 1 - xs$, $M = -1$, $\xi = 1$, then $u(x) = [(-5)^{\frac{1}{2}} - 2] + 3x$ and $\bar{u}(x) = -[(-5)^{\frac{1}{2}} + 2] + 3x$ are conjugate imaginary characteristic functions corresponding respectively to the conjugate imaginary characteristic numbers $\lambda = [(-5)^{\frac{1}{2}} + 2]/6$, $\bar{\lambda} = -[(-5)^{\frac{1}{2}} - 2]/6$.

We adopt the notation

$$\int^* g(s)\,ds = Mg(\xi) + \int g(s)\,ds,$$

thus the transformation (1) and integral equation (2) are now written

$$Tf(x) = \int^* K(x,s)f(s)\,ds$$

$$\lambda u(x) = \int^* K(x,s)u(s)\,ds.$$

The operator $\int^*$ we name the *M-integral*, and for convenience we name the corresponding operation *M-integration*. The M-integral is a special instance of the Stieltjes integral, and we shall assume its elementary properties throughout this paper without enumerating them.

In Section I we begin by defining an M-orthonormed system of functions, that is, orthonormed with respect to the M-integral, and which reduces, for $M = 0$, to an ordinary orthonormed system. In laying down this definition allowance must be made for the fact that the M-integral of a positive function may be negative, and it is on account of this property that we are unable to generalize the fundamental inequalities of Bessel and Schwarz. In the same section we establish the existence of a complete M-orthonormed system.

The complete M-orthonormed system forms the starting point for the solution of our problem. We use it to pass from the integral equation (2), following the classic method of Hilbert, to a system of equations in infinitely many variables, which has been solved by A. Pell-Wheeler.[5] This furnishes a proof of the existence of characteristic numbers for the integral equation, and also an expansion theorem. The results are given in Section II.

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[5] A. Pell-Wheeler, "Spectral theory for a certain class of non-symmetric completely continuous matrices," *American Journal of Mathematics*, vol. 57 (1935), pp. 847-853.

SECTION I. *M*-ORTHONORMED SYSTEMS.

1. *Preliminary definitions.* The functions

$$\phi_1(x), \phi_2(x), \cdots, \phi_n(x), \cdots,$$

finite or infinite in number, form an *M-orthogonal system* if they satisfy the conditions

$$\int^* \phi_i(x)\phi_j(x)\,dx = 0, \qquad i \neq j.$$

They form an *M-orthonormed system* if

$$\int^* \phi_i(x)\phi_j(x)\,dx = \pm e_{ij}, \qquad e_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

A system of functions $\{\phi_n(x)\}$, continuous on I, is *M-closed* if there exists no function $f(x)$, continuous on I, which is M-orthogonal to all the functions of the system.

An M-orthonormed system of functions $\{\phi_n(x)\}$, continuous on I, is *M-complete* if, for all possible functions $f(x)$ continuous on I, the functions ϕ_n of the system satisfy the relation

$$\sum_{n=1}^{\infty} \pm \left(\int^* f\phi_n\right)^2 = \int^* f^2,$$

the negative sign in the summation occurring when and only when the value of n is such that $\int^* \phi_n^2 = -1$.

2. *Construction of an M-orthonormed system.* Let $\{f_n(x)\}$ be a set of functions, infinite in number, which satisfy the conditions

(a) They are continuous on I.

(b) They are linearly independent on I.

(c) At least one function of the set does not have the value zero at $x = \xi$.

(d) $D_n(f) \neq 0, \qquad (n = 1, 2, \cdots),$

where $D_n(f)$ is the n-rowed determinant with general element $\int^* f_i f_j$ $(i, j = 1, 2, \cdots, n)$.

From this set of functions we build up by the method of determinants an M-orthonormed system $\{\phi_n(x)\}$ as follows:

$$\text{(1a)} \qquad \begin{cases} \phi_1 = f_1 \Big/ \sqrt{\pm \int^* f_1^2} \\ \phi_n = E_n(f) / \sqrt{\pm D_{n-1}(f) \cdot D_n(f)}, \qquad (n = 2, 3, \cdots), \end{cases}$$

where $E_n(f)$ is the determinant derived from $D_n(f)$ by replacing the elements of the n-th column, reading downwards, by $f_1, f_2, \cdots, f_n$. The positive (negative) sign is to be used if the product of the determinants under the radical sign is positive (negative), and in the case of $n = 1$ if $\int^* f_1^2$ is positive (negative).

It can be easily verified that the system $\{\phi_n\}$ thus defined satisfies the definition of an M-orthonormed system. Each ϕ_n is expressed linearly and homogeneously, with constant coefficients, in terms of the functions $f_1, f_2, \cdots, f_n$, and conversely.

The functions ϕ_n are real and continuous on I. They are furthermore linearly independent there; for, suppose there exist constants c_n, not all zero, such that

$$\sum_{n=1}^{p} c_n \phi_n \equiv 0;$$

multiplication throughout by ϕ_n and M-integration over I leads to $c_n = 0$ $(n = 1, 2, \cdots, p)$, a contradiction.

Conversely, from an M-orthonormed system of continuous functions $\{\phi_n\}$ it is possible to construct an orthonormed set $\{f_n\}$ of the form

$$f_n = \sum_{p=1}^{n} c_p \phi_p$$

where the c_p are constants, not all zero. This property is an immediate consequence of the linear independence of the ϕ_n, just proved.

The f_n may be transformed into an orthonormed system of functions which satisfy the four conditions specified above, and since such a transformation simplifies considerably the expressions for ϕ_n we shall assume it has been previously carried through; then formula (1a) reduces, after expansion of the determinants to the alternative form

$$\text{(1b)} \qquad \begin{cases} \phi_1 = \dfrac{f_1}{\sqrt{\pm (1 + M f^2_{1,\xi})}} \\ \phi_n = \dfrac{f_n (1 + M \sum_{i=1}^{n-1} f^2_{i,\xi}) - M f_{n,\xi} \sum_{i=1}^{n-1} f_{i,\xi} f_i}{\sqrt{\pm (1 + M \sum_{i=1}^{n-1} f^2_{i,\xi})(1 + M \sum_{i=1}^{n} f^2_{i,\xi})}}, \quad (n = 2, 3, \cdots). \end{cases}$$

The subscript ξ indicates that the value of the function is to be taken at $x = \xi$.

It is desirable at this point to introduce the following abbreviations:

$$A_p(x) = \sum_{i=1}^{p} f_{i,\xi} f_i(x), \qquad A_0(x) \equiv 0,$$

$$P_p = (1 + MA_{p-1,\xi})(1 + MA_{p,\xi}), \qquad (p = 1, 2, \cdots).$$

Then formula (1a) assumes the final abridged form

$$\phi_n = \frac{f_n(1 + MA_{n-1,\xi}) - Mf_{n,\xi}A_{n-1}}{\sqrt{\pm P_n}}, \qquad (n = 1, 2, \cdots). \tag{1c}$$

We are able to make certain general deductions concerning the sign of P_n. There are two possibilities:

(a) P_n is positive for all n. Then $1 + MA_{n,\xi}$ is also, since otherwise P_1 would be negative. Whence

$$A_{n,\xi} < -1/M, \qquad (n = 1, 2, \cdots).$$

Since $A_{n,\xi}$ is monotonic, increasing and bounded, $\lim_{n=\infty} A_{n,\xi}$ exists and is finite. This is a necessary condition that $P_n > 0$ $(n = 1, 2, \cdots)$.

(b) P_n is positive for all values of n except one, say N, which satisfies the inequality

$$-1/A_{N-1,\xi} < M < -1/A_{N,\xi}, \qquad N > 1$$

or

$$M < -1/A_{N,\xi}, \qquad N = 1.$$

In this case P_N is negative. A sufficient condition for the existence of such a value N of n is that $\lim_{n=\infty} A_{n,\xi} = \infty$.

P_n cannot be negative for more than one value of n under any circumstances, for if N is the first value of n for which $P_n < 0$, we have

$$1 + MA_{N,\xi} < 0,$$

$$\therefore \quad 1 + MA_{n,\xi} < 0, \qquad n > N,$$

$$\therefore \quad P_n = (1 + MA_{n-1,\xi})(1 + MA_{n,\xi}) > 0, \qquad n > N.$$

3. *M-completeness of the system* (1c). From the foregoing results we obtain

THEOREM I. *If the functions f_n satisfy the four conditions of § 2, then the set $\{\phi_n\}$ given by* (1c) *can contain at most one function satisfying the relation*

$$\int^* \phi_i^2 = -1.$$

In order to facilitate the proof of the next theorem we first introduce the

LEMMA. *If the set $\{f_n\}$ is complete, then a necessary and sufficient condition that*

$$\lim_{n=\infty} \frac{\left(\int fA_n - f_\xi\right)^2}{1 + MA_{n,\xi}} = 0 \tag{2}$$

is that $\lim_{n=\infty} A_{n,\xi} = \infty$, where f is an arbitrary function continuous on I.

(a) *Necessary condition.* From the Lagrange-Cauchy inequality we find

$$\left| \int fA_n \right| \leq \sum_{i=1}^{n} \left| f_{i,\xi} \int ff_i \right| \leq \sqrt{A_{n,\xi}} \cdot \sqrt{\sum_{i=1}^{n} \left(\int ff_i \right)^2}$$

for all n, and all continuous functions f, and therefore

$$\sqrt{A_{n,\xi}} \geq \left| \int fA_n \right| \Big/ \sqrt{\sum_{i=1}^{n} \left(\int ff_i \right)^2}.$$

If $\lim_{n=\infty} A_{n,\xi}$ is finite, then from (2), $\lim_{n=\infty} \int fA_n = f_\xi$ for all f under consideration, and hence

$$\lim_{n=\infty} \left| \int fA_n \right| \Big/ \sqrt{\sum_{i=1}^{n} \left(\int ff_i \right)} = |f_\xi| \Big/ \sqrt{\int f^2}.$$

For a suitable choice of f this limit can be made greater than any arbitrarily assigned positive number,[6] which contradicts the assumption that $\lim_{n=\infty} A_{n,\xi}$ be finite.

---

[6] For example, define $f(x) = F_n^{\,2}(x)$ as follows:

$$\begin{aligned} F_n^{\,2}(x) &= 0, && 0 \leq x \leq (n-1)\xi/n \\ &= nx/\xi + (1-n) && (n-1)\xi/n \leq x \leq \xi \\ &= nx/(\xi-1) + (\xi - n\xi - 1)/(\xi - 1) && \xi \leq x \leq (n\xi + 1 - \xi)/n \\ &= 0 && (n\xi + 1 - \xi)/n \leq x \leq 1, \end{aligned}$$

$\xi \neq 0$ or 1. As n increases, $\int f^2$ decreases, but $f(\xi) = 1$ for all n. If $\xi = 0$ or 1 this definition needs to be modified somewhat.

(b) *Sufficient condition.* If $\int fA_n$ approaches a finite limit with increasing n, the sufficiency of the condition is self-evident; it remains then, to investigate the case where this limit is infinite. To this end it suffices to prove that

$$\lim_{n=\infty} \int fA_n/\sqrt{A_{n,\xi}} = 0.$$

This fraction may be written in the form

$$\int fA_n/\sqrt{A_{n,\xi}} = \sum_{i=1}^{m} f_{i,\xi} \int ff_i/\sqrt{A_{n,\xi}} + \sum_{i=m+1}^{n} f_{i,\xi} \int ff_i/\sqrt{A_{n,\xi}}, \qquad n > m.$$

Application of the Lagrange-Cauchy inequality to the second term on the right gives

$$\left| \sum_{i=m+1}^{n} f_{i,\xi} \int ff_i/\sqrt{A_{n,\xi}} \right| \leq \left[\sqrt{\sum_{i=m+1}^{n} f^2{}_{i,\xi}}/\sqrt{A_{n,\xi}} \right] \cdot \sqrt{\sum_{i=m+1}^{n} \left(\int ff_i \right)^2}$$

$$\leq 1 \cdot \epsilon/2 \text{ for } m > N_1,\ n > m.$$

If we take a fixed value for $m > N_1$, then

$$\left| \sum_{i=1}^{m} f_{i,\xi} \int ff_i/\sqrt{A_{n,\xi}} \right| \leq \left[\max |f_{i,\xi}| \cdot \max \left| \int ff_i \right| /\sqrt{A_{n,\xi}} \right] \cdot m, \qquad (i = 1, 2, \cdots, m).$$

$$\leq \epsilon/2 \text{ for } n > N_2 > N_1.$$

It follows that

$$\left| \int fA_n/\sqrt{A_{n,\xi}} \right| = \left| \sum_{i=1}^{n} f_{i,\xi} \int ff_i/\sqrt{A_{n,\xi}} \right| < \epsilon, \qquad n \geq N_2,$$

which completes the proof of the Lemma.

THEOREM II. *If the set $\{f_n\}$ of § 2 is complete, then a necessary and sufficient condition that the system $\{\phi_n\}$ defined by* (1c) *be M-complete is that*

$$\lim_{n=\infty} A_{n,\xi} = \infty.$$

If we substitute the given expression for ϕ_n in the series

$$\sum_{n=1}^{\infty} \pm \left(\int^{*} f\phi_n \right)^2$$

we get

$$\sum_{n=1}^{\infty} \pm \left(\int^* f\phi_n \right)^2 = \sum_{n=1}^{\infty} \Big[(1 + MA_{n-1,\xi}) \int^* ff_n - Mf_{n,\xi} \int^* fA_{n-1} \Big]^2 / P_n$$

$$= \sum_{n=1}^{\infty} \Big[M^2 f^2_{n,\xi} f\xi^2 + 2Mf_{n,\xi} f_\xi \Big\{ (1 + MA_{n-1,\xi}) \int ff_n - Mf_{n,\xi} \int fA_{n-1} \Big\}$$

$$+ \Big\{ (1 + MA_{n-1,\xi}) \int ff_n - Mf_{n,\xi} \int fA_n \Big]^2 / P_n.$$

The sum S_p of the first p terms of this series is

$$S_p = \Big[M^2 A_{p,\xi} f\xi^2 + 2Mf_\xi \int fA_p + \sum_{i=1}^{p} \left(\int ff_i \right)^2 (1 + MA_{p,\xi} - Mf^2{}_{i,\xi})$$

$$- 2M \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \left(f_{i,\xi} \int ff_i \right) \left(f_{j,\xi} \int ff_j \right) \Big] / (1 + MA_{p,\xi}).$$

This expression can be simplified by a suitable combination of terms to give

$$S_p = \frac{-M \left(\int fA_p - f_\xi \right)^2}{1 + MA_{p,\xi}} + \sum_{i=1}^{p} \left(\int ff_i \right)^2 + Mf_\xi^2.$$

If we now take the limit of both sides of this equation and apply the Lemma and the hypothesis on the system $\{f_n\}$ we obtain the required result:

$$\lim_{n=\infty} S_p = \sum_{n=1}^{\infty} \pm \left(\int^* f\phi_n \right)^2 = \sum_{i=1}^{\infty} \left(\int ff_i \right)^2 + Mf_\xi^2$$

$$= \int f^2 + Mf_\xi^2 = \int^* f^2.$$

THEOREM III. *If the systems $\{\phi_n\}$ and $\{f_n\}$ are M-complete and complete respectively, then $\{\phi_n\}$ is M-closed.*

To prove this theorem we assume the existence of a continuous function $\psi(x)$ which is M-orthogonal to all the ϕ_n. It is accordingly M-orthogonal to all the f_n, i. e.

$$\int^* \psi f_n = \int \psi f_n + M\psi_\xi f_{n,\xi} = 0, \qquad (n = 1, 2, \cdots).$$

If either or both of M and ψ_ξ are zero then $\psi \equiv 0$. If not, we have

$$\sum_{n=1}^{\infty} \left(\int \psi f_n \right)^2 = M^2 \psi_\xi^2 \sum_{n=1}^{\infty} f^2{}_{n,\xi}.$$

The series on the left is convergent, that on the right divergent by Theorem II. Hence the assumption is incorrect and $\{\phi_n\}$ is M-closed.

4. *The case of zero denominators.* The investigations so far have proceeded on the assumption that none of the denominators of (1a), or, what comes to the same thing, none of the factors P_n of (1c), is zero. Obviously P_n may be zero for one or more values of n. We now show that this contingency may be overcome by redefining the functions ϕ_n in terms of the f_n.

We first replace all conditions made hitherto on these f_n by the following:

(a) They are continuous on I.

(b) They are orthonormed on I.

(c) The system $\{f_n\}$ is complete.

(d) $\lim_{n=\infty} A_{n,\xi} = \infty$.

If the first denominator in (1c) to vanish is that of ϕ_m, then

$$1 + MA_{m,\xi} = 0.$$

Thus $f_{m,\xi} \neq 0$, but any or all of $f_{i,\xi}$ $(1 \leq i \leq m-1)$ may be zero. Also by condition (d) above, there exists a function f_i $(i > m)$ of the set $\{f_n\}$ such that $f_{i,\xi} \neq 0$. Let f_{m+q} $(q \geq 1)$ be the first such function of the set, i. e.

$$f_{i,\xi} = 0 \quad (m < i < m+q), \qquad f_{(m+q),\xi} \neq 0.$$

Then the set of functions ϕ_n $(1 \leq n < m)$ whose values are given by (1c), § 2, and the functions

$$\phi_n = f_{n+1} \qquad (m \leq n \leq m+q-2)$$

are M-orthonormed on I. For each ϕ_n of this set P_n is positive.

If we express ϕ_{m+q-1} and ϕ_{m+q} each as linear combinations of the $m+q$ functions f_i $(1 \leq i \leq m+q)$, and impose on them the conditions of M-orthonormality, we arrive at the expressions

$$\phi_{m+q-1} = r \pm s$$
$$\phi_{m+q} = r \mp s$$

where

$$r = [(f^2_{m,\xi} + f^2_{(m+q),\xi}) \sum_{i=1}^{m-1} f_{i,\xi} f_i + (f^2_{m,\xi} - f^2_{(m+q),\xi}) c f_{m,\xi} f_m + 2c^2 f^2_{m,\xi} f_{(m+q),\xi} f_{m+q}] / 2c f^2_{m,\xi} f_{(m+q),\xi}$$

$$s = [f^2_{m,\xi} \sum_{i=1}^{m-1} f_{i,\xi} f_i + f^3_{m,\xi} f_m] / 2c f^2_{m,\xi} f_{(m+q),\xi}.$$

The constant c may not have the value zero, but otherwise it is arbitrary; for simplicity it will be taken equal to unity. Also

$$\int^* \phi^2_{m+q-1} = -\int^* \phi^2_{m+q} = \pm 1.$$

If we now take for ϕ_n $(n > m+q)$ the values given by (1c), § 2, the system $\{\phi_n\}$ thus constructed is M-orthonormed on I, and can have no zero denominators.

Moreover $\int^* \phi_n{}^2 = +1$ for all values of n except one, either $n = m+q-1$ or $n = m+q$.

Theorems I, II and III are valid for the system $\{\phi_n\}$ just obtained. We present here the proof of Theorem II only, as the others offer no difficulty.

If S_n represents the sum of the first n terms of the series

$$\sum_{i=1}^{\infty} \pm \left(\int^* f\phi_i\right)^2 \tag{3}$$

then

$$S_{m+q+p} - S_{m+q} = \sum_{i=m+q+1}^{m+q+p} f^2{}_{i,\xi} \left(\int fA_{m+q} - f_\xi\right)^2 \Big/ f^2{}_{(m+q),\xi} \sum_{i=m+q}^{m+q+p} f^2{}_{i,\xi}$$
$$+ \sum_{i=m+q+1}^{m+q+p} \left(\int ff_i\right)^2 + I_p, \qquad p \geq 1,$$

where

$$I_p = \left[2f_\xi \sum_{i=m+q+1}^{m+q+p} f_{i,\xi} \int ff_i - \left(\sum_{i=m+q+1}^{m+q+p} f_{i,\xi} \int ff_i\right)^2\right] \Big/ \sum_{i=m+q}^{m+q+p} f^2{}_{i,\xi}.$$

Since $\lim_{p=\infty} I_p = 0$ [7] we have

$$\lim_{p=\infty} (S_{m+q+p} - S_{m+q}) = \left(\int fA_{m+q} - f_\xi\right)^2 \Big/ f^2{}_{(m+q),\xi} + \sum_{i=m+q+1}^{\infty} \left(\int ff_i\right)^2.$$

The sum of the first $m+q$ terms of the series (3) is given by

$$S_{m+q} = \left[\int fA_{m-1} - f_\xi\right]^2 \Big/ f^2{}_{m,\xi} + \sum_{i=1}^{m-1} \left(\int ff_i\right)^2 + Mf_\xi{}^2$$
$$+ \sum_{i=m+1}^{m+q-1} \left(\int ff_i\right)^2 + \Big[\left(f^2{}_{m,\xi} + f^2{}_{(m+q),\xi}\right)\left(\int fA_{m-1} - f_\xi\right)^2$$
$$- 2f^2{}_{m,\xi}\left(f_{m,\xi} \int ff_m + f_{(m+q),\xi} \int ff_{m+q}\right)\left(\int fA_{m-1} - f_\xi\right)$$
$$- f^2{}_{m,\xi}\,(f^2{}_{m,\xi} - f^2{}_{(m+q),\xi})\left(\int ff_m\right)^2$$
$$- 2f^3{}_{m,\xi} f_{(m+q),\xi} \int ff_m \int ff_{m+q}\Big] \Big/ f^2{}_{m,\xi} f^2{}_{(m+q),\xi}.$$

[7] Cf. proof of Lemma, § 3.

Combining these results, we have after cancellation

$$\sum_{n=1}^{\infty} \pm \left(\int^* f\phi_n \right)^2 = \int f^2 + Mf\xi^2 = \int^* f^2.$$

We are now justified in stating that if a set of functions $\{f_n\}$ exists which satisfies the four conditions enumerated above, then it is always possible to construct from it an M-complete, M-closed system $\{\phi_n\}$ of continuous functions which are linear combinations of the f_n, of the form

$$\phi_n = \sum_{i=1}^{p_n} c_{ni} f_i.$$

An example of such a set $\{f_n\}$ is given by the sequence of trigonometric functions

$$\{f_n\} = 1,\ 2^{\frac{1}{2}} \cos 2\pi x,\ 2^{\frac{1}{2}} \sin 2\pi x,\ 2^{\frac{1}{2}} \cos 4\pi x,\ 2^{\frac{1}{2}} \sin 4\pi x,\ \cdots.$$

In concluding this section we assemble the salient facts obtained so far in the

THEOREM IV. *For every M there exist M-complete, M-closed systems $\{\phi_n\}$ of functions which are continuous and M-orthogonal on I, and such that $\int^* \phi_n^2 = \pm 1$, where the negative sign occurs for one and only one value of n.*

SECTION II. EQUATION WITH SYMMETRIC KERNEL.

1. *Preliminary definitions and propositions.* We turn now to the equation

$$\lambda u(x) = \int^* K(x,s)u(s)\,ds, \tag{1}$$

in which the kernel $K(x,s)$ is a continuous symmetric [8] function on R.

By a *characteristic number* of equation (1) is understood a value of the parameter λ for which the equation has a continuous solution $u(x)$ not identically zero on I.

In a similar fashion we employ the ordinary definitions of *characteristic function, pole of the resolvent, order of a pole,* etc., with the understanding that they refer to equation (1) unless definitely stated otherwise.

LEMMA. *If $K(x,s)$* is symmetric on R, then

$$\int^* \int^* f(x)K(x,s)g(s)\,dsdx = \int^* \int^* f(s)K(s,x)g(x)\,dsdx$$

for all continuous functions f and g, and conversely.

[8] $K(x,s)$ is symmetric on R if $K(x,s) \equiv K(s,x)$ on R, otherwise it is non-symmetric.

The first statement in the Lemma is readily verified. To prove the converse we assume that $K(x,s)$ is not symmetric, and arrive at a contradiction.

Application of the hypothesis to the equality gives

$$\int^* f(x)\left[\int^* \{K(x,s)-K(s,x)\}g(s)\,ds\right]dx = 0.$$

Let $\{\phi_n(x)\}$ be an M-closed system of continuous functions. If we take $f(x)$ equal, sequentially, to these functions ϕ_n, then

$$\int^* \phi_n(x)\left[\int^* \{K(x,s)-K(s,x)\}g(s)\,ds\right]dx = 0, \quad (n=1,2,\cdots).$$

This is impossible, in view of the closure of the system $\{\phi_n\}$, unless

$$\int^* \{K(x,s)-K(s,x)\}g(s)\,ds \equiv 0.$$

If we take $g(s)$ equal, sequentially, to the same functions $\phi_n(s)$, and regard x as a parameter, then a similar process of reasoning gives

$$K(x,s)-K(s,x) \equiv 0 \text{ on } R,$$

i. e., $K(x,s)$ is symmetric on R.

2. *Properties of characteristic numbers.*

THEOREM I. *If the kernel $K(x,s)$ of (1) is symmetric on R, and if $u_1(x)$, $u_2(x)$ are characteristic functions of $K(x,s)$ for λ_1, λ_2 respectively, $(\lambda_1 \neq \lambda_2)$, then $u_1(x)$ and $u_2(x)$ are M-orthogonal on I.*

For, following Poisson's method of proof for the ordinary integral equation, we obtain

$$\lambda_1 \int^* u_1(x)u_2(x)\,dx = \int^*\int^* u_2(x)K(x,s)u_1(s)\,ds\,dx$$

$$\lambda_2 \int^* u_1(x)u_2(x)\,dx = \int^*\int^* u_1(x)K(x,s)u_2(s)\,ds\,dx,$$

hence, if we subtract these and apply the preceding lemma we get

$$\lambda_1 \int^* u_1(x)u_2(x)\,dx = \lambda_2 \int^* u_1(x)u_2(x)\,dx, \qquad \lambda_1 \neq \lambda_2,$$

and from this follows

$$\int^* u_1(x)u_2(x)\,dx = 0.$$

THEOREM II. *A sufficient condition that all of the characteristic numbers of equation* (1) *for a symmetric kernel* $K(x,s)$ *be real is that* $K(x,s)$ *satisfy the inequality*

$$\int^* \int^* f(x)K(x,s)f(s)\,ds\,dx < 0 \text{ or } > 0\,^{9}$$

for all continuous functions $f \not\equiv 0$ *on* I.

Assume that $u(x) = v(x) + iw(x)$ is a complex characteristic function corresponding to $\lambda \neq 0$. Multiply (1) throughout by the conjugate characteristic function $\bar{u}(x) = v(x) - iw(x)$ and M-integrate with respect to x. We get

$$\begin{aligned}\lambda \int^* \bar{u}(x)u(x)\,dx &= \int^* \int^* \bar{u}(x)K(x,s)u(s)\,ds\,dx \\ &= \int\int v(x)K(x,s)v(s)\,ds\,dx + \int\int w(x)K(x,s)w(s)\,ds\,dx \\ &\quad + 2M[v(\xi)\int K(x,\xi)v(x)\,dx + w(\xi)\int K(x,\xi)w(x)\,dx] \\ &\quad + M^2K(\xi,\xi)[v^2(\xi) + w^2(\xi)].\end{aligned}$$

Since $\int^* \bar{u}(x)u(x)\,dx$ is real, and the right-hand side is real, a sufficient condition that λ be real is that $\int^* \bar{u}u \not\equiv 0$, or that the expression on the right be different from zero. This is satisfied if

$$\int^* \int^* fKf > 0 \text{ or } < 0$$

for all continuous functions $f \not\equiv 0$ on I, which was to be proved.

COROLLARY I. *If in the pure integral equation*

$$\mu\phi(x) = \int K(x,s)\phi(s)\,ds \tag{2}$$

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[9] For example, if the system $\{\phi_n\}$ is M-closed on I, and if

$$K(x,s) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(s)}{\eta_n^2},$$

the η_n^2 so chosen that the series converges uniformly on I, then

$$\int^* \int^* fKf = \sum_{n=1}^{\infty} \frac{1}{\eta_n^2}\left(\int^* f(x)\phi_n(x)\,dx\right)^2 \geq 0,$$

the equality sign holding only if $f \equiv 0$.

the kernel $K(x, s)$ is positive,[10] *and if the characteristic functions ϕ_α of* (2) *form a complete orthonormed system for this $K(x, s)$, and are such that*

$$\lim_{n=\infty} \sum_{\alpha=1}^{n} \phi^2_{\alpha,\xi} = \infty,$$

then the characteristic numbers of (1) *for the same kernel are real.*

The proof is based upon two theorems of Mercer's [10] by which we may write

$$K(x, s) = \sum_{\alpha=1}^{\infty} \mu_\alpha \phi_\alpha(x) \phi_\alpha(s),$$

the characteristic numbers μ_α of (2) being all positive. Then

$$\int^* \int^* fKf = \sum_{\alpha=1}^{\infty} \mu_\alpha \left(\int^* f\phi_\alpha \right)^2.$$

The series on the right may be zero only if $f \equiv 0$, since otherwise

$$\int^* f\phi_\alpha = \int f\phi_\alpha + Mf_\xi \phi_{\alpha,\xi} = 0, \qquad (\alpha = 1, 2, \cdots),$$

$$\therefore \quad \sum_{\alpha=1}^{\infty} \left(\int f\phi_\alpha \right)^2 = M^2 f_\xi{}^2 \sum_{\alpha=1}^{\infty} \phi^2_{\alpha,\xi};$$

the series on the left of the last equation is convergent, that on the right divergent, hence

$$\int^* \int^* fKf \geq 0,$$

the equality holding only if $f \equiv 0$; therefore the characteristic numbers of (1) are real by the previous theorem.

COROLLARY 2. *A sufficient condition that all of the characteristic numbers of equation* (1) *for a symmetric kernel $K(x, s)$ be real is that $K(x, s)$ satisfy the inequality*

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[10] We use here Mercer's definition of a positive function: A symmetric function $K(x, s)$ is positive if

$$\int \int fKf \geq 0$$

for all continuous functions f. See *Phil. Trans. Royal Soc. A*, vol. 209 (1909), pp. 417-444.

$$\int\int [K(\xi, x)K(\xi, s) - K(\xi, \xi)K(x, s)]f(x)f(s)\,ds\,dx < 0$$

for all continuous functions $f \not\equiv 0$.[11]

For the inequality of Theorem II is true for all $f(\xi)$ if its discriminant is negative, i. e., if

$$M^2\Big[\int K(x, \xi)f(x)\,dx\Big]^2 - K(\xi, \xi)M^2 \int\int f(x)K(x, s)f(s)\,ds\,dx < 0,$$

hence if

$$\int\int [K(x, \xi)K(\xi, s) - K(\xi, \xi)K(x, s)]f(x)f(s)\,ds\,dx < 0$$

for all continuous $f \not\equiv 0$.

THEOREM III. EXISTENCE THEOREM. *Every symmetric kernel has at least one characteristic number.*

To facilitate the proof we first introduce the following definition:

A matrix $A = (a_{ij})$ is *M-symmetric* if it is symmetric except for one row, say the N-th, and its corresponding column, and if the elements a_{Ni} and a_{iN} of this row and column satisfy the relations

$$a_{Ni} = -a_{iN}, \qquad i \neq N.$$

As was stated in the Introduction we use Hilbert's method [12] for passing from an integral equation to a system of linear equations in infinitely many unknowns.

Let $\{\phi_n\}$ be an M-complete system of the type defined by formula (1c), Section I. Multiply equation (1) throughout by $\phi_i(x)$, M-integrate with respect to x, and apply Theorem II,[13] Section I; this gives

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[11] An example of such a kernel is

$$K(x, s) = 1 + \sum_{n=1}^{\infty} \frac{\sin n(x-s) \cdot \sin n(x-\xi)}{\eta_n^2},$$

since

$$-\int\int [K(\xi, x)K(\xi, s) - K(\xi, \xi)K(x, s)]f(x)f(s)\,ds\,dx$$

$$= -\sum_{n=1}^{\infty} \frac{1}{\eta_n^2}\left(\int f(x) \sin n(x-\xi)\right)^2 \qquad < 0 \text{ for } f \not\equiv 0.$$

[12] Hilbert, *Grundzüge einer allgemeinen Theorie der linearen integral-Gleichungen*, pp. 186-188.

[13] In Section I we assumed implicitly that the arbitrary function f was real. There is nothing in the proof of Theorem II, Section I, which will not be valid if f is a continuous complex function.

$$\lambda \int^* \phi_i u = \int^* \int^* \phi_i K u$$

$$= \sum_{K=1}^{\infty} \pm \int^* \int^* \phi_i K \phi_K \int^* u \phi_K, \qquad (i = 1, 2, \cdots).$$

If we introduce the substitutions

$$\int^* \phi_i u = x_i, \qquad \pm \int^* \int^* \phi_i K \phi_K = a_{iK}$$

the above equations are replaced by the linear system

$$\lambda x_i = \sum_{K=1}^{\infty} a_{iK} x_K, \qquad (i = 1, 2, \cdots), \tag{3}$$

in the infinitely many unknowns x_i.

If we let the x_i represent the components of a vector x in a space of infinitely many dimensions, then x is a vector of the Hilbert space,[14] for

$$\sum_{i=1}^{\infty} \pm x_i \bar{x}_i = \sum_{i=1}^{\infty} \pm \int^* u \phi_i \int^* \bar{u} \phi_i = \int^* u \bar{u},$$

hence

$$\sum_{i=1}^{\infty} x_i \bar{x}_i \text{ converges.}$$

The matrix of (3) is M-symmetric, and it is completely continuous.[15] To prove the latter property it is sufficient to demonstrate that $\sum_{iK=1}^{\infty} a^2_{iK}$ is convergent.

By definition

$$\sum_{iK=1}^{\infty} a^2_{iK} = \sum_{iK=1}^{\infty} \left[\int^* \int^* \phi_i K \phi_K \right]^2.$$

If Theorem II, Section I is applied twice to the series on the right of this equality we obtain successively

$$\sum_{K=1}^{\infty} \left[\int^* \int^* \phi_i K \phi_K \right]^2 = \int^* \left[\int^* \phi_i K \right]^2 + 2 \left[\int^* \int^* \phi_i K \phi_N \right]^2$$

$$\sum_{iK=1}^{\infty} \left[\int^* \int^* \phi_i K \phi_K \right]^2 = \sum_{i=1}^{\infty} \int^* \left[\int^* \phi_i K \right]^2 + 2 \left[\int^* K \phi_N \right]^2$$

$$+ 4 \left[\int^* \int^* \phi_N K \phi_N \right]^2.$$

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[14] A vector $x = (x_i)$ is said to belong to the Hilbert space if $\sum_{i=1}^{\infty} x_i \bar{x}_i$ converges.

[15] Hilbert, *Grundzüge*, Kap. XI, p. 147 f. The convergence of $\sum_{iK=1}^{\infty} a^2_{iK}$ is a necessary and sufficient condition for the complete continuity of the matrix A.

The series $\sum_{i=1}^{\infty}\left[\int^* \phi_i K\right]^2$ is uniformly convergent, since it converges to the continuous function

$$\int^* K^2 + 2\left[\int^* K\phi_N\right]^2$$

and since the terms of the series are positive.[16]

Application of Theorem II, § 3 of Section I to this series yields

$$\sum_{i=1}^{\infty}\int^*\left[\int^* \phi_i K\right]^2 = \int^*\sum_{i=1}^{\infty}\left[\int^* \phi_i K\right]^2 = \int^*\int^* K^2 + 2\int^*\left[\int^* K\phi_N\right]^2,$$

hence $\sum_{iK=1}^{\infty} a^2{}_{iK}$ is convergent.

It has been proved [17] that a system of equations of the type (3), the matrix of which is M-symmetric, has at least one characteristic number. Let x then, be a characteristic vector of (3) belonging to the Hilbert space. The series

$$\sum_{K=1}^{\infty} \pm x_K \int^* K\phi_K \tag{4}$$

is absolutely and uniformly convergent on I. This follows at once if we apply the Lagrange-Cauchy inequality to the series obtained from (4) by omitting those terms, at most two in number,[18] which correspond to complex values of λ. Denote the sum of the series by $\lambda u(x)$:

$$\lambda u(x) = \sum_{K=1}^{\infty} \pm x_K \int^* K\phi_K,$$

multiply this equation throughout by $\phi_i(x)$, and M-integrate with respect to x; then

$$\lambda \int^* \phi_i u = \sum_{K=1}^{\infty} \pm x_K \int^*\int^* \phi_i K\phi_K.$$

Since, by hypothesis, x is a characteristic vector of the system of linear equations (3) we have

$$x_i = \int^* \phi_i u,$$

and therefore

$$\lambda u(x) = \sum_{K=1}^{\infty} \pm \int^* u\phi_K \int^* K(x,s)\phi_K(s)\,ds$$
$$= -\int^* K(x,s)u(s)\,ds.$$

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[16] E. W. Hobson, *The Theory of Functions of a Real Variable*, Sec. Ed., vol. 2, p. 116.

[17] A. Pell-Wheeler, *loc. cit.*, Theorem II.

[18] Cf. p. 878.

Thus a unique correspondence has been established between characteristic vectors of (3) and characteristic functions of the integral equation (1). The symmetric kernel $K(x, s)$ of (1) has therefore at least one characteristic number, which was to be proved.

In the Introduction examples were given of kernels wi[t]h ..ero and imaginary characteristic numbers, and of characteristic numbers [w]hich are not simple poles of the resolvent. There are however, certain restri[c]tions that can be made on them.

It can easily be proved [19] that one characteristic numb[er] may be zero, that two may be imaginary (conjugate), the rest being real, that a characteristic number can not be a pole of the resolvent of order greater than three, and that there can not be more than one which is not a simple pole of the resolvent. A necessary condition that λ be not a simple pole of the resolvent of equation (1) is that $\int^* u^2 = 0$.[20]

3. *Expansion theorem.* Our problem is this: to determine sufficient conditions upon a continuous function $g(x)$ in order that it may be expressible in the form

$$g(x) = \sum_a c_a u_a(x),$$

where the u_a are the characteristic functions of the integral equation (1) for the symmetric kernel $K(x, s)$.

Certain functions $U_a(x)$ [21] will be introduced corresponding to a characteristic number λ_m which is not a simple pole of the resolvent. These functions satisfy the relations

$$\int^* U_a U_\beta = \pm e_{a\beta}$$

$$\int^* KU_a = \sum_\beta \lambda_m c_{a\beta} U_\beta, \quad |c_{a\beta}| \neq 0, \qquad \lambda_m \neq 0,$$

$$\int^* KU_a = \sum_\beta c_{a\beta} U_\beta, \quad |c_{a\beta}| = 0, \qquad \lambda_m = 0.$$

A. Pell-Wheeler has proved [19] that the iterated matrix $A^{(4)} = (a_{iK}^{(4)})$ is developable into the following series:

$$a_{iK}^{(4)} = \sum_{j=1}^{4} E_j^{(4)}, \tag{5}$$

[19] A. Pell-Wheeler, *loc. cit.*, Theorem I.

[20] That the condition is not sufficient is illustrated by Example 1, Introduction.

[21] These functions U_a are analogous to the so-called "principal functions" of the Fredholm theory.

where $E_1^{(4)}$ is that part of the expansion corresponding to real characteristic numbers λ_a which are simple poles of the resolvent, and is an absolutely convergent series; $E_2^{(4)}$ contains those terms corresponding to imaginary λ; $E_3^{(4)}$ those terms corresponding to $\lambda \neq 0$ not a simple pole of the resolvent; and $E_4^{(4)}$ those terms corresponding to $\lambda = 0$ not a simple pole of the resolvent.

The expressions written out in full for $E_j^{(4)}$ are

$$E_1^{(4)} = \sum_a \pm \lambda_a{}^4 s_{ai} s^*{}_{aK},$$

the s_a being characteristic vectors for the matrix A, and also $A^{(4)}$, and satisfying the relations

$$(s_a s^*{}_\beta) = \sum_i s_{ai} s^*{}_{\beta i} = \pm e_{a\beta};$$

$$E_2^{(4)} = \pm (\lambda^4 s_i s^*{}_K + \lambda^4 \bar{s}_i \bar{s}^*{}_K),$$

where

$$(s\bar{s}^*) = 0, \qquad (ss^*) = \pm 1;$$

$$E_3^{(4)} = \lambda_m{}^4 \sum_{\alpha\beta} \pm c_{\alpha\beta} s_{\alpha i} s^*{}_{\beta K}, \qquad |c_{\alpha\beta}| \neq 0, \qquad (\alpha, \beta = 1, 2 \text{ or } 1, 2, 3),$$

where

$$(s_a s^*{}_\beta) = \pm e_{a\beta}, \quad \sum_K a_{iK}^{(4)} s_{aK} = \lambda_m{}^4 \sum_\beta c_{a\beta} s_{\beta K};$$

$$E_4^{(4)} = \sum_{\alpha\beta} \pm c_{\alpha\beta} s_{\alpha i} s^*{}_{\beta K}, \qquad |c_{\alpha\beta}| = 0, \qquad (\alpha, \beta = 1, 2 \text{ or } 1, 2, 3),$$

where

$$(s_a s^*{}_\beta) = \pm e_{a\beta}, \qquad \sum_K a_{iK}^{(4)} s_{aK} = \sum_\beta c_{a\beta} s_{\beta K}.$$

From (5) we have

$$\sum_{iK} a_{iK}^{(4)} x_i y_K = \sum_{iK} [\sum_j E_j] x_i y_K, \tag{6}$$

the series being absolutely and uniformly convergent for all vectors x and y of the Hilbert space. This condition is satisfied if we take

$$x_i = \int^* f\phi_i \quad \text{and} \quad y_i = \int^* g\phi_i,$$

the functions f and g being continuous on I. Substitute these values in (6), and the expression on the left becomes

$$\sum_{iK} a_{iK}^{(4)} x_i y_K = \sum_{iK} \pm \int^* \int^* \phi_i K^{(4)} \phi_K \int^* f\phi_i \int^* g\phi_K$$

$$= \int^* \int^* K^{(4)} fg,$$

by Theorem II, Section I.

If we make the same substitutions in the right side of (6) we obtain

$$\sum_{iK} E_1^{(4)} x_i y_K = \sum_a \pm \lambda_a^4 \sum_i \pm \int^* u_a \phi_i \int^* f \phi_i \sum_K \pm \int^* u_a \phi_K \int^* g \phi_K$$
$$= \sum_a \pm \lambda_a^4 \int^* u_a f \int^* u_a g,$$

the series being absolutely and uniformly convergent;

$$\sum_{iK} E_2^{(4)} x_i y_K = \pm \Big[\lambda^4 \sum_i \pm \int^* u \phi_i \int^* f \phi_i \sum_K \int^* u \phi_K \int^* g \phi_K$$
$$+ \bar{\lambda}^4 \sum_i \pm \int^* \bar{u} \phi_i \int^* f \phi_i \sum_k \int^* \bar{u} \phi_K \int^* g \phi_K \Big]$$
$$= \pm \Big(\lambda^4 \int^* uf \int^* ug + \bar{\lambda}^4 \int^* \bar{u} f \int^* \bar{u} g \Big);$$

and the remaining terms of (6), corresponding to a characteristic number not a simple pole of the resolvent, are respectively

$$\sum_{iK} E_3^{(4)} x_i y_K = \lambda_m^4 \sum_{a\beta} \pm c_{a\beta} \sum_i s_{ai} x_i \sum_K s^*_{\beta K} y_K, \qquad |c_{a\beta}| \neq 0,$$

$$\sum_{iK} E_4^{(4)} x_i y_K = \sum_{a\beta} \pm c_{a\beta} \sum_i s_{ai} x_i \sum_K s^*_{\beta K} y_K, \qquad |c_{a\beta}| = 0.$$

The vectors s_a appearng in these two expressions are equal to the same linear combinations of characteristic vectors for λ_m as the functions U_a, introduced above, are of the characteristic functions of (1), corresponding to the same λ_m; and the conditions imposed upon the s_a lead, under the transformation

$$s_{ai} = \int^* U_a \phi_i$$

to the conditions imposed upon the U_a. The converse is also true. Hence

$$\sum_{iK} E_3^{(4)} x_i y_K = \lambda_m^4 \sum_{a\beta} \pm c_{a\beta} \int^* U_a f \int^* U_\beta g, \qquad |c_{a\beta}| \neq 0,$$

$$\sum_{iK} E_4^{(4)} x_i y_K = \sum_{a\beta} \pm c_{a\beta} \int^* U_a f \int^* U_\beta g, \qquad |c_{a\beta}| = 0.$$

Combining these results, we have

$$(7) \qquad \int^* \int^* K^{(4)} fg = \sum_a \pm \lambda_a^4 \int^* u_a f \int^* u_a g$$
$$\pm \Big(\lambda^4 \int^* uf \int^* ug + \bar{\lambda}^4 \int^* \bar{u} f \int^* \bar{u} g \Big)$$
$$+ \lambda_m^4 \sum_{a\beta} \pm c_{a\beta} \int^* U_a f \int^* U_\beta g$$
$$+ \sum_{a\beta} \pm c_{a\beta} \int^* U_a f \int^* U_\beta g.$$

Since the series

$$\sum_a \pm \lambda_a^4 \left(\int^* u_a v \right)^2$$

converges for all continuous functions $v(x)$, then the series

$$\sum_a \pm \lambda_a^6 u_a^2(x),$$

and hence the series

$$\sum_a \pm \lambda_a^5 u_a \int^* u_a v$$

converges absolutely and uniformly.

Set

$$(8) \quad h(x) = \int^* K^{(5)} f - \Big[\sum_a \lambda_a^4 u_a \int^* \int^* f K^{(4)} u_a \pm \Big(\lambda^1 u \int^* \int^* f K^{(4)} u + \bar{\lambda}^4 \bar{u} \int^* \int^* f K^{(4)} \bar{u} \Big) + \sum_{a\beta} \pm \lambda_m^4 c_{a\beta} U_a \int^* \int^* f K^{(4)} U_\beta + \sum_{a\beta} \pm c_{a\beta} U_a \int^* \int^* f K^{(4)} U_\beta \Big].$$

If l is an arbitrary continuous function, then

$$\int^* hl = \int^* \int^* l K^{(5)} f - \Big[\sum_a \lambda_a^4 \int^* l u_a \int^* \int^* f K^{(4)} u_a \pm \Big(\lambda^4 \int^* l u \int^* \int^* f K^{(4)} u + \bar{\lambda}^4 \int^* l \bar{u} \int^* \int^* f K^{(4)} \bar{u} \Big) + \sum_{a\beta} \pm \lambda_m^4 c_{a\beta} \int^* l U_a \int^* \int^* f K^{(4)} U_\beta + \sum_{a\beta} \pm c_{a\beta} \int^* l U_a \int^* \int^* f K^{(4)} U_\beta.$$

Since the right side of this equality is zero from (7), it follows that $h \equiv 0$ by the Lemma, Section II.

If we now choose g such that

$$g(x) = \int^* K^{(5)} f$$

and make the substitution in (8), we have an expansion of g in the form

$$g(x) = \sum_a \pm u_a \int^* g u_a \pm \Big(u_a \int^* g u_a + \bar{u}_a \int^* g \bar{u}_a \Big) + \sum_a \pm U_a \int^* g U_a.$$

In conclusion we set forth the

THEOREM IV. *Every function of the form*

$$g(x) = \int^* K^{(5)}(x, s) f(s)\, ds,$$

where $f(s)$ is an arbitrary function continuous on I, can be expressed in the form of an absolutely and uniformly convergent series:

$$g(x) = \sum_a \pm u_a(x) \int^* . g u_a + E(x),$$

where the u_a are characteristic functions of the integral equation (1), *and $E(x)$ consists of a finite number of terms arising from the existence of characteristic numbers which are conjugate imaginary, or zero, or are poles of the resolvent of order greater than one.*

BRYN MAWR COLLEGE.

ON THE EXPANSION OF HARMONIC FUNCTIONS IN SERIES OF HARMONIC POLYNOMIALS BELONGING TO A SIMPLY CONNECTED REGION.[1]

By O. J. FARRELL.

1. *Introduction.* In his address [2] to the American Mathematical Society on the approximation of harmonic functions by harmonic polynomials and by harmonic rational functions Professor Walsh included a treatment of expansions in given regions in terms of a particular set of harmonic polynomials belonging to that region. Among the results reported in this connection was a theorem of his own to the effect that *if C be a simple finite analytic curve in the (x, y)-plane, then there exist harmonic polynomials $\{p_n(x, y)\}$ such that if $f(x, y)$ is defined and continuous on C and on C is of bounded variation then $f(x, y)$ can be developed into a series*

$$\text{(a)} \qquad f(x, y) = \sum_{k=1}^{\infty} a_k p_k(x, y)$$

which converges uniformly in the closed interior of C. Series (a) *thus represents a function harmonic interior to C, continuous in the corresponding closed region, and having the value $f(x, y)$ on C. There exist continuous functions $\{q_n(x, y)\}$ on C with which the polynomials $\{p_n(x, y)\}$ form a biorthogonal set:*

$$\int_C p_k(x, y) q_m(x, y)\, ds = \begin{cases} 0, & \text{if } k \neq m \\ 1, & \text{if } k = m. \end{cases}$$

The coefficients of (a) *are given by the formulas*

$$a_k = \int_C f(x, y) q_k(x, y)\, ds.$$

The functions $\{q_n(x, y)\}$ depend on C but not on $f(x, y)$.

If the curve C of this theorem is the unit circle, the situation is classical; and if polar coördinates (ρ, ϕ) be introduced in the (x, y)-plane, the functions $\{p_n(x, y)\}$ may be chosen directly as the set of harmonic polynomials $\{\rho^n \cos n\phi, \rho^n \sin n\phi\}$. For the Fourier development of $f(x, y)$ on the unit

[1] Presented to the American Mathematical Society, December 27, 1929 and September 7, 1934.

[2] Walsh, *Bulletin of the American Mathematical Society*, vol. 35 (1929), pp. 499-544.

circle gives rise at once to a series in terms of these functions, and this series converges uniformly throughout the closed interior of the unit circle.

The report of the theorem just quoted was followed by mention of the fact that the theorem would lend itself readily to generalizations. It is the purpose of the present paper to contribute one such generalization as follows:

THEOREM I. *In the (x, y)-plane let G be a limited simply connected region whose boundary B consists wholly of simple*[3] *boundary points and is also the boundary of an infinite region. Let the region G be mapped conformally onto the interior of the unit circle in the w-plane, $w = x' + iy' = \rho e^{i\phi}$, by means of an analytic function $w = \Phi(z)$, $z = x + iy$, and denote by C_μ the transform in the (x, y)-plane of the circle $\rho = \mu$ in the (x', y')-plane. Then there exist harmonic polynomials $\{p_n(x, y)\}$ and a series of functions $\{q_n(x, y)\}$ continuous and biorthogonal on every curve C_μ, $0 < \mu < 1$, to the polynomials $\{p_n(x, y)\}$:*

$$\int_{C_\mu} p_k(x, y) q_m(x, y)\, ds = \begin{cases} 0, & \text{if } k \neq m \\ 1, & \text{if } k = m. \end{cases}$$

If $f(x, y)$ is an arbitrary function harmonic in G, and if C_μ be the transform of an arbitrarily chosen circle $\rho = \mu$, $0 < \mu < 1$, then the series

$$(1) \qquad \sum_{k=1}^{\infty} a_k p_k(x, y), \qquad a_k = \int_{C_\mu} f(x, y) q_k(x, y)\, ds,$$

converges to $f(x, y)$ continuously[4] *in G. If $f(x, y)$ is harmonic interior to C_μ, then the formal expansion (1) of $f(x, y)$ found by integration over $C_{\mu'}$, $0 < \mu' < \mu$, converges to $f(x, y)$ continuously in the region interior to C_μ. If $f(x, y)$ is merely known to be defined and continuous on C_μ, then the formal expansion (1) of $f(x, y)$ found by integration on C_μ is summable (C_1) to $f(x, y)$ uniformly on C_μ, hence summable (C_1) uniformly on and within C_μ, thus furnishing a solution of the Dirichlet problem for the region interior to C_μ and the boundary values $f(x, y)$.*

If the region G of Theorem I happens to be the interior of the unit circle, the situation is again the classical one, and the functions $\{p_n(x, y)\}$ may be

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[3] A boundary point is simple if it is contained in just one primend. See Carathéodory, *Mathematische Annalen*, vol. 73 (1913), pp. 321-370, §§ 44-46.

[4] A series is said to converge continuously in a region if in any closed subregion the series converges uniformly. See Walsh, *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 668-689, § 2.

taken as the harmonic polynomials $\{\rho^n \cos n\phi, \rho^n \sin n\phi\}$. The curves C_μ are now the circles $\rho = \mu$, $0 < \mu < 1$; and the biorthogonal q-functions are the functions $\{(1/\pi\rho^{n+1}) \cos n\phi, (1/\pi\rho^{n+1}) \sin n\phi\}$.

To obtain Theorem I we first consider (in Theorems II and III below) expansions of functions harmonic within the unit circle in the (x', y')-plane. These expansions are not in terms of harmonic polynomials but in terms of a set of harmonic functions $\{p'_n(x', y')\}$ which approximate to the functions $\{(1/\pi^{\frac{1}{2}})\rho^n \cos n\phi, (1/\pi^{\frac{1}{2}})\rho^n \sin n\phi\}$. If the approximations be taken sufficiently close, an arbitrary function $f'(x', y')$ harmonic within the unit circle can be expanded into a series in terms of the functions $\{p'_n(x', y')\}$ where the coefficients of the expansion are found with the help of a biorthogonal set of functions by integration over an arbitrary circle $\gamma_\mu : \rho = \mu$, $0 < \mu < 1$. The region G of Theorem I is then mapped onto the interior of the unit circle in the (x', y')-plane of Theorem III and by means of the properties of the conformal map the result of Theorem III yields its analog in the (x, y)-plane and gives us Theorem I.

2. *Developments within the unit circle.* The point of departure here is the following specialization of a theorem of Walsh:[5]

THEOREM II. *Let the functions* $\{\pi_n(w)\}$, $w = x' + iy' = \rho e^{i\phi}$, *be analytic for* $|w| \leqq 1 + \epsilon$, $\epsilon > 0$, *and such that for* $|w| \leqq 1 + \epsilon$ *we have*

$$|\pi_n(w) - (1/\pi^{\frac{1}{2}})w^n| \leqq \bar{\epsilon}_n, \qquad (n = 1, 2, \cdots), \tag{2}$$

where the series $\Sigma\bar{\epsilon}_n$ *converges and the sum of the series* $\Sigma\bar{\epsilon}_n^2$ *is less than* $1/4\pi$. *Let*

$$\begin{aligned} \pi_0(w) &= p'_1(x', y') = 1/(2\pi)^{\frac{1}{2}} \\ \pi_n(w) &= p'_{2n}(x', y') + ip'_{2n+1}(x', y'), \qquad (n = 1, 2, \cdots). \end{aligned} \tag{3}$$

Then there exists a set of functions $\{q'_n(x', y')\}$ *defined and continuous on the unit circle* $\gamma : \rho = 1$ *and such that*

$$\int_\gamma p'_k(x', y') q'_m(x', y')\, d\phi = \begin{cases} 0, & \text{if } k \neq m \\ 1, & \text{if } k = m. \end{cases}$$

Any function $f'(x', y')$ *which is continuous and of bounded variation on* γ *can be developed into a series*

$$f'(x', y') = \sum_{k=1}^{\infty} a_k p'_k(x', y'), \qquad a_k = \int_\gamma f'(x', y') q'_k(x', y')\, d\phi, \tag{4}$$

[5] *Proceedings of the National Academy of Sciences*, vol. 13 (1927), pp. 175-180, Theorem 3.

which converges uniformly throughout the closed interior of γ, thus defining a function harmonic for $\rho < 1$, continuous for $\rho \leqq 1$, and equal to $f'(x', y')$ on γ. If $f'(x', y')$ is merely known to be continuous on γ, then the formal development (4) of $f'(x', y')$ on γ is summable (C_1) to $f'(x', y')$ uniformly on γ, hence summable (C_1) uniformly on and within γ, thus furnishing a solution of the Dirichlet problem for the region interior to γ and the boundary values $f'(x', y')$.

It is seen at once that Theorem II differs from the result as given by Walsh only in that the functions $\{p'_n(x', y')\}$ have been specialized. For if $\pi_n(w)$ is analytic in the closed interior of the circle $\rho = 1 + \epsilon$, the real and imaginary parts of $\pi_n(w)$ are harmonic in this closed region. And inequalities (2) imply the original inequalities (5) of Walsh's Theorem 3. For it follows from (2) that

$$\begin{aligned} &| p'_{2n}(x', y') - (1/\pi^{\frac{1}{2}})\rho^n \cos n\phi | \leqq \bar{\epsilon}_n \\ &| p'_{2n+1}(x', y') - (1/\pi^{\frac{1}{2}})\rho^n \sin n\phi | \leqq \bar{\epsilon}_n \qquad (n = 1, 2, \cdots). \end{aligned}$$

And if the $\bar{\epsilon}_n$ of (2) be identified with the ϵ_n required by Walsh by setting $\bar{\epsilon}_n = \epsilon_{2n} = \epsilon_{2n+1}$ $(n = 1, 2, \cdots)$, we have, since $\epsilon_1 = 0$,

$$\sum_{n=1}^{\infty} \epsilon_n^2 = \sum_{n=2}^{\infty} \epsilon_n^2 = 2 \sum_{n=1}^{\infty} \bar{\epsilon}_n^2 < 1/2\pi.$$

Theorem II will now be applied to prove the following theorem.

THEOREM III. *For $| w | < 1$ let the functions $\pi_n(w)$ $(n = 1, 2, \cdots)$, be analytic and such that*

$$| \pi_n(w) - (1/\pi^{\frac{1}{2}})w^n | \leqq \lambda^n \bar{\epsilon}_n \qquad (n = 1, 2, \cdots), \tag{5}$$

where λ is any fixed positive number less than unity, and where the $\bar{\epsilon}_n$ satisfy the hypotheses of Theorem II. Suppose further that $\pi_n(w)$ has a zero of order n at the origin. Let the functions $\{p'_n(x', y')\}$ be defined as in (3) of Theorem II. Then there exists a set of functions $\{q''_n(x', y')\}$ which are continuous and biorthogonal to the set $\{p'_n(x', y')\}$ on every circle $\gamma_\mu : \rho = \mu$, $0 < \mu < 1$:

$$\int_{\gamma_\mu} p'_k(x', y') q''_m(x', y') \, d\sigma = \begin{cases} 0, & \text{if } k \neq m \\ 1, & \text{if } k = m, \end{cases}$$

where σ denotes arc length along γ_μ. If $f'(x', y')$ is any function harmonic within the unit circle γ, then the series

$$\sum_{k=1}^{\infty} a_k p'_k(x', y'), \qquad a_k = \int_{\gamma_\mu} f'(x', y') q''_k(x' y') \, d\sigma, \tag{6}$$

converges to $f'(x', y')$ continuously thruout the interior of γ. If $f'(x', y')$ is harmonic interior to γ_μ, then the formal development (6) *of $f'(x', y')$ found by integration over $\gamma_{\mu'}$, $0 < \mu' < \mu$, converges to $f'(x', y')$ continuously interior to γ_μ. If $f'(x', y')$ is merely known to be defined and continuous on γ_μ, then the formal expansion* (6) *of $f'(x', y')$ on γ_μ is summable (C_1) to $f'(x', y')$ uniformly on γ_μ hence is summable (C_1) uniformly on and within γ_μ, thus furnishing a solution of the Dirichlet problem for the region interior to γ_μ and the boundary values $f'(x', y')$.*

To prove this theorem we shall show first that if μ be chosen arbitrarily between zero and unity, there is a positive ϵ' such that

$$(7) \qquad | \pi_n(\mu w)/\mu^n - (1/\pi^{\frac{1}{2}}) w^n | \leqq \bar{\epsilon}_n, \qquad | w | < 1 + \epsilon'.$$

From (5) it follows that

$$| \pi_n(w)/w^n - 1/\pi^{\frac{1}{2}} | \leqq \lambda^n \bar{\epsilon}_n / | w |^n, \qquad w \neq 0, \qquad | w | < 1.$$

Moreover, since $\pi_n(w)$ has a zero of order n at the origin, the function $g_n(w) = \pi_n(w)/w^n - 1/\pi^{\frac{1}{2}}$ when properly defined at the origin is analytic everywhere within the unit circle. The inequalities (5) may be written

$$| w^n g_n(w) | / \lambda^n \bar{\epsilon}_n \leqq 1, \qquad | w | < 1,$$

whence by Schwarz's Lemma

$$| w^n g_n(w) | / \lambda^n \bar{\epsilon}_n \leqq | w |, \qquad | w | < 1,$$

that is

$$| w^{n-1} g_n(w) | / \lambda^n \bar{\epsilon}_n \leqq 1, \qquad | w | < 1.$$

By repeated use of Schwarz's Lemma we finally get

$$| g_n(w) | \leqq \lambda^n \bar{\epsilon}_n, \qquad | w | < 1.$$

Since this last inequality holds for all w of modulus less than unity, it will hold when w is replaced by μw, $0 < \mu < 1$. Accordingly we have

$$| \pi_n(\mu w)/\mu^n w^n - 1/\pi^{\frac{1}{2}} | \leqq \lambda^n \bar{\epsilon}_n, \qquad \mu | w | < 1,$$

or

$$| \pi_n(\mu w)/\mu^n - (1/\pi^{\frac{1}{2}}) w^n | \leqq \lambda^n \bar{\epsilon}_n | w |^n, \qquad \mu | w | < 1.$$

The quantity $\lambda^n \bar{\epsilon}_n | w |^n$ is not greater than $\bar{\epsilon}_n$ if $| w | < 1/\lambda$. If then we choose a positive ϵ' such that $1 + \epsilon'$ is less than the smaller of $1/\lambda$ and $1/\mu$, we have the desired inequalities (7).

Let $f'(x', y')$ be a function harmonic within the unit circle. Choose a number μ between zero and unity. By Theorem II the function $f'(\mu x', \mu y')$ admits an expansion

$$(8) \qquad f'(\mu x', \mu y') = b_1 p'_1(\mu x', \mu y') + \cdots + b_{2n} p'_{2n}(\mu x', \mu y')/\mu^n + b_{2n+1} p'_{2n+1}(\mu x', \mu y')/\mu^n + \cdots,$$

which converges uniformly in the closed interior of γ. Moreover, there exists a set of functions $\{\bar{q}_n(x', y')\}$ continuous and biorthogonal on γ to the set

$$p'_1(\mu x', \mu y'), \cdots, p'_{2n}(\mu x', \mu y')/\mu^n, \; p'_{2n+1}(\mu x', \mu y')/\mu^n, \cdots,$$

and the coefficients of (8) are given by

$$b_k = \int_\gamma f'(\mu x', \mu y') \bar{q}_k(x', y') \, d\phi.$$

But (8) is equivalent to a development of $f'(x', y')$ converging uniformly on and within γ_μ, namely,

$$(9) \qquad f'(x', y') = \sum_{k=1}^{\infty} a_k p'_k(x', y'), \qquad a_k = \int_{\gamma_\mu} f'(x', y') q_k^{(\mu)}(x', y') \, d\sigma,$$

where the functions $\{q_n^{(\mu)}(x', y')\}$ are defined by the relations

$$q_{2n}^{(\mu)}(x', y') = \bar{q}_{2n}(x'/\mu, y'/\mu)/\mu^{n+1}, \qquad q_{2n+1}^{(\mu)}(x', y') = \bar{q}_{2n+1}(x'/\mu, y'/\mu)/\mu^{n+1}.$$

Hence for each value of μ, $0 < \mu < 1$, there is a set of functions $\{q_n^{(\mu)}(x', y')\}$ continuous and biorthogonal to the set $\{p'_n(x', y')\}$ on the circle γ_μ. And if we let $q''_n(x', y')$ be identified on each circle γ_μ with $q_n^{(\mu)}(x', y')$, we obtain a set of functions $\{q''_n(x', y')\}$ continuous and biorthogonal to the set $\{p'_n(x', y')\}$ on every circle $\gamma_\mu : \rho = \mu$, $0 < \mu < 1$.

We have yet to show that the expansion (6) formed for a chosen value of μ converges continuously thruout the interior of γ. Let S be any closed subregion of the interior of γ. There is some number ν, $0 < \nu < 1$, such that both S and γ_μ are interior to the circle $\gamma_\nu : \rho = \nu$. Then $f'(x', y')$ admits an expansion

$$(10) \qquad f'(x', y') = \sum_{k=1}^{\infty} g_k p'_k(x', y'), \qquad g_k = \int_{\gamma_\nu} f'(x', y') q''_k(x', y') \, d\sigma,$$

which converges uniformly on and within γ_ν and hence uniformly in S and on γ_μ. But (10) must be the same development as (6). For multiplication of these two series thru by $q''_k(x', y') \, d\sigma$ and integration term by term over γ_μ gives the equality of a_k and g_k.

This completes the proof of Theorem III for the case where $f'(x', y')$ is harmonic interior to the unit circle. The remaining two cases are handled in similar manner.

3. *Proof of Theorem I.* An arbitrary region of the type described in Theorem I can be mapped conformally onto the interior of the unit circle in the w-plane of Theorem III by means of analytic functions $w = \Phi(z)$ and $z = \Psi(w)$. The function $w = \Phi(z)$ may be defined on the boundary B of G so as to be continuous in the closed region $G + B$. We shall suppose that this has been done. Moreover each of the functions

$$w^n/\pi^{\frac{1}{2}} = [\Phi(z)]^n/\pi^{\frac{1}{2}} \qquad (n = 1, 2, \cdots),$$

can be approximated uniformly in the closed region $G + B$ as closely as desired by a polynomial $P_n(z)$,[6] and when this approximation is taken sufficiently close $P_n(z)$ will have by Rouché's theorem [7] precisely n zeroes in G. Moreover, these n zeroes may be made to coincide [8] at the point of G which corresponds by the map to the origin in the (x', y')-plane. Let the polynomials $\{P_n(z)\}$ be so chosen and also so that

$$| P_n(z) - [\Phi(z)]^n/\pi^{\frac{1}{2}} | \leqq \lambda^n \bar{\epsilon}_n, \qquad z \text{ in } G + B,$$

where $\bar{\epsilon}_n$ are those of Theorem III. Consequently inequalities (5) will hold for $| w | < 1$, if we identify the functions $\{\pi_n(w)\}$ with the transforms of the polynomials $\{P_n(z)\}$. Accordingly the harmonic polynomials $\{p_n(x, y)\}$ are to be chosen as the set of real and imaginary parts of the polynomials $\{P_n(z)\}$ so that their transforms may be identified with the harmonic functions $\{p'_n(x', y')\}$ of Theorem III.

Theorem I now follows from Theorem III by virtue of the conformal map. For if $f(x, y)$ is harmonic in G, its transform $f'(x', y')$ is harmonic within the unit circle γ, and so $f'(x', y')$ may be expanded into a series in terms of the

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[6] Farrell, *American Journal of Mathematics*, vol. 54 (1932), pp. 571-578, Lemma in § 5.

[7] See Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 1, p. 185. The application of Rouché's theorem here is almost immediate. The function $\Omega_n(z) = (1/\pi^{\frac{1}{2}})[\Phi(z)]^n$ is certainly zero nowhere on the boundary of G. And if the approximating polynomial $P_n(z)$ be chosen so that $| P_n(z) - \Omega_n(z) | < 1/\pi^{\frac{1}{2}}$ for z on B, then we have $| P_n(z) - \Omega_n(z) | < | \Omega_n(z) |$ for z on B. It follows then by the theorem of Rouché that the two functions $\Omega_n(z)$ and $\Omega_n(z) + [P_n(z) - \Omega_n(z)]$ have the same number of zeroes in G, which means that $P_n(z)$ has just n zeroes in G.

[8] See Walsh, *Transactions of the American Mathematical Society*, vol. 26 (1924), pp. 155-170, p. 164.

functions $\{p'_n(x', y')\}$ where the coefficients of the expansion are found by integration around an arbitrary circle γ_μ. The series converges continuously within γ. This means that $f(x, y)$ may be expanded into a series in terms of the harmonic polynomials $\{p_n(x, y)\}$, where the series converges continuously in G. In like manner the remaining two cases of Theorem I are readily verified. The functions $\{q_n(x, y)\}$ of Theorem I are to be defined by the relations

$$q_n(x, y) = q''_n(x', y')\, d\sigma/ds,$$

where the equation refers to points of C_μ and γ_μ which correspond under the conformal map, and where σ and s denote arc length along γ_μ and C_μ respectively.

UNION COLLEGE,
SCHENECTADY, NEW YORK.

CERTAIN PROBLEMS IN THE THEORY OF CLOSEST APPROXIMATION.[1]

By PAUL G. HOEL.

Introduction. This paper is concerned with properties of minimizing sums for integrals containing the m-th power of the error of an approximation, primarily for values of $m < 1$. By a minimizing sum is meant a linear combination of a set of n suitably restricted, linearly independent functions $s_1(x), s_2(x), \cdots, s_n(x)$ which minimizes the integral

$$\int_a^b w(x) \mid f(x) - \psi(x) \mid^m dx, \tag{1}$$

where $w(x)$ is a suitably restricted weight function, m is a given positive constant, $\psi(x) = c_1 s_1(x) + c_2 s_2(x) + \cdots + c_n s_n(x)$ is an arbitrary linear combination of the s's, and $f(x)$ is a given suitably restricted function not identically equal to a linear combination of the s's, to which it is desired to approximate by means of the ψ's.

Three properties of minimizing sums are considered here. The first two are extensions, with appropriate modifications in statement, of properties of minimizing polynomials which are well known for an exponent $m > 1$, while the discussion of the third is believed to be new for $m \geq 1$, as well as for $m < 1$. The paper is divided into three sections corresponding to these three properties.

Throughout this paper, except where explicitly stated otherwise, the function $f(x)$ and the properly independent functions $s_1(x), s_2(x), \cdots, s_n(x)$ are assumed to be bounded and measurable on the interval (a, b), and the weight function $w(x)$ to be summable and non-negative on (a, b), but positive over a subset of positive measure. The functions $s_1(x), s_2(x), \cdots, s_n(x)$ are said to be properly independent on (a, b) if every linear combination of them, in which the coefficients are not all zero, is different from zero on a subset of positive measure. Under these restrictions, it can be shown by the usual methods of proof [2] that there exists at least one linear combination of the s's which minimizes (1). Let $\phi_m(x)$ denote such a minimizing sum. For $m < 1$ there is not in general a unique minimizing sum; therefore, it will be assumed that $\phi_m(x)$ represents any one of the minimizing sums, if more than one exists.

[1] Presented to the American Mathematical Society, June 23, 1933.

[2] See, for example, D. Jackson, "A generalized problem in weighted approximation," *Transactions of the American Mathematical Society*, vol. 26 (1924), pp. 133-154; p. 137.

1. *The orthogonality property.* The purpose of this section is to demonstrate that the orthogonality property of a minimizing sum holds, within certain restrictions, when the exponent $m \leq 1$. In the case of $m > 1$, D. Jackson [8] has shown that this property is both necessary and sufficient for a continuous minimizing sum of a continuous function. Here the condition will be shown to be a necessary one for this more general class of minimizing sums and functions, but naturally under more severe restrictions when $m \leq 1$.

THEOREM 1. *For $m < 1$ the expression $w(x)\dfrac{|f(x)-\phi_m(x)|^m}{f(x)-\phi_m(x)}$, provided that both it and the expression with $w(x)$ deleted are summable on (a, b), is orthogonal over the interval (a, b) to every linear combination $\psi(x)$ of the s's; that is,*

$$\int_a^b w(x)\frac{|f(x)-\phi_m(x)|^m}{f(x)-\phi_m(x)}\psi(x)\,dx = 0.$$

Proof. Let $I(h) = \int_a^b w\,|r-hs|^m\,dx$, where $r \equiv f-\phi_m$, s is any one of the s's, and h is a constant. Then $I(0)$ represents the minimized integral. Without restricting the problem, it may be assumed that $|s(x)| \leq 1$, for the assumption can be realized by defining a new set of s's which are the proper constant multiples of the old set. Now form

$$\frac{I(h)-I(0)}{h} = \int_a^b w\,\frac{|r-hs|^m-|r|^m}{h}\,dx. \tag{2}$$

Let E_1, E_2, and E_3 be the subsets of (a, b) for which respectively $r > 0$, $r < 0$, and $r = 0$. Then the derivative of $I(h)$ at the point $h = 0$ will exist and will be given by

$$I'(0) = \lim_{h=0}\frac{I(h)-I(0)}{h} = \lim_{h=0}\int_{E_1+E_2+E_3} w\,\frac{|r-hs|^m-|r|^m}{h}\,dx, \tag{3}$$

if the limit exists for the integrals over E_1, E_2, and E_3 separately. Only the limit as h approaches zero through positive values will be discussed, but the method is identical for the limit from the left.

From the hypothesis of the theorem that $|r|^m/r$ is summable on (a, b), it follows that the measure of the set E_3 is zero. Consequently, the integral over E_3 vanishes and its limit is zero.

In considering the integral over E_1, let $h_1, h_2, \cdots, h_i, \cdots$ be an arbitrary set of positive numbers for which $\lim_{i\to\infty} h_i = 0$. Let $g_1, g_2, \cdots, g_i, \cdots$ denote

[8] "On functions of closest approximation," *ibid.*, vol. 22 (1921), pp. 117-128; p. 126.

the corresponding values of the integrand in (3). This set of functions is a sequence which satisfies the following three conditions almost everywhere on E_1.

(a) For every value of h_i, $g_i(x)$ is obviously summable on E_1.

(b) $\lim_{i\to\infty} g_i(x) = g(x)$, if, by definition, $g(x) \equiv -w(x)ms(x)[r(x)]^{m-1}$.

For, if x_0 represents an arbitrary point of E_1, it is possible, from the definition of the set E_1, to select an h_i so small that $h_i < r(x_0)/2$. Then for $h \leq h_i$, since $|s(x)| \leq 1$, $|r(x_0) - hs(x_0)|^m = [r(x_0) - hs(x_0)]^m$ is a positive differentiable function of h; consequently,

$$\lim_{h=0} \frac{|r(x_0) - hs(x_0)|^m - |r(x_0)|^m}{h} = \left\{ \frac{d}{dh}[r(x_0) - hs(x_0)]^m \right\}_{h=0} = -ms(x_0)[r(x_0)]^{m-1}.$$

This limit holds in particular as h approaches zero through the succession of values h_i; therefore,

$$\lim_{i\to\infty} g_i(x_0) = -w(x_0)ms(x_0)[r(x_0)]^{m-1} = g(x_0).$$

(c) $|g_i(x)| \leq G(x)$, if, by definition, $G(x) \equiv 10w(x)[r(x)]^{m-1}$. For, when $r(x) > 2h_i$ and $h \leq h_i$, the law of the mean may be applied to the expression below, considered as a differentiable function of h, to give

$$(4) \quad |r - hs|^m - |r|^m = [r - hs]^m - [r]^m = -msh[r - \theta_1 hs]^{m-1}, \quad 0 < \theta_1 < 1.$$

Furthermore, $[r - \theta_1 h_i s]^{m-1} < [r/2]^{m-1} < 2[r]^{m-1}$. Consequently, because of (4)

$$|g_i(x)| = |wms[r - \theta_1 h_i s]^{m-1}| \leq 2mw|s|[r]^{m-1} \leq 2w[r]^{m-1}.$$

When $r(x) \leq 2h_i$,

$$|g_i(x)| = w\left|\frac{|r - h_i s|^m - |r|^m}{h_i}\right| \leq w\frac{(3h_i)^m + (2h_i)^m}{h_i} \leq 10w[r]^{m-1}.$$

Hence (c) holds for all x and i, where $G(x)$, by hypothesis, is summable on E_1.

Now, by a well known theorem on Lebesgue integration, the fact that $\{g_i(x)\}$ satisfies the above three conditions justifies the limit

$$(5) \quad \lim_{i\to\infty} \int_{E_1} w\frac{|r - h_i s|^m - |r|^m}{h_i}\,dx = -m\int_{E_1} ws[r]^{m-1}\,dx.$$

Since $\{h_i\}$ is an arbitrary sequence with limit zero, (5) implies that

$$\lim_{h=0} \int_{E_1} w \frac{|r-hs|^m - |r|^m}{h}\, dx = -m \int_{E_1} ws[r]^{m-1}\, dx.$$

The methods used to obtain this limit may be applied to the integral in (3) over E_2 to obtain a similar result. Combination of these three limit results gives,

$$I'(0) = -m \int_{E_1} ws \frac{|r|^m}{r}\, dx - m \int_{E_2} ws \frac{|r|^m}{r}\, dx = -m \int_a^b ws \frac{|r|^m}{r}\, dx.$$

Now $I(h)$ is a continuous function of h which has a minimum at $h=0$; and since it possesses a derivative at that point, the derivative must vanish there. Since s represents any one of the s's, this means that

$$\int_a^b ws_i \frac{|r|^m}{r}\, dx = 0 \qquad (i = 1, 2, \cdots, n),$$

which is equivalent to the vanishing integral of the theorem.

For $m=1$ the details of the proof are simpler while the theorem hypotheses merely require that the measure of E_3 be zero.

If $w(x)$ is positive on (a, b), in addition to its original restrictions, Theorem 1 will be found to hold without the summability restriction on $|r|^m/r$.

It should be noted in the case of polynomial approximation that when the orthogonality property holds, it gives at once the information that $r(x)$ must possess at least n sign-changes in (a, b), since otherwise $\psi(x)$ could be chosen as a polynomial of degree $\leq n-1$ with the same sign as $|r|^m/r$ to contradict the theorem. With n replaced by $2n+1$, the same statement can be made concerning trigonometric approximation.

2. *Zero multiplicity bounds of the error function $f(x) - \phi_m(x)$.* This section is concerned *with polynomial approximation to a continuous function.* When $f(x)$ is continuous and $m > 1$, it is known [4] that the error function corresponding to a minimizing polynomial of degree $n-1$ changes sign at least n times in the interval. When $f(x)$ is continuous and $m=1$, it is known [5] that the error function changes sign at least n times or else vanishes over a set of positive measure. When $f(x)$ is analytic and $m < 1$, it is known [6] that the sum of the multiplicities of the zeros of the error function must be at

[4] *Loc. cit.*, "On functions of closest approximation."

[5] D. Jackson, "Note on a class of polynomials of approximation," *Transactions of the American Mathematical Society*, vol. 22 (1921), pp. 320-326.

[6] D. Jackson, "Note on the convergence of a sequence of approximating polynomials," *Bulletin of the American Mathematical Society*, vol. 37 (1931), pp. 69-72; p. 72.

least n. The problem here is to make as definite a statement as possible when $m < 1$ and $f(x)$ is assumed to be merely continuous.

In this section let $w(x)$ be bounded, in addition to previous restrictions. As before let $r(x) = f(x) - \phi_m(x)$ denote a minimizing error function, where now $\phi_m(x)$ represents a minimizing polynomial of degree $\leq n-1$. Since $r(x)$ need not have zeros of definite multiplicities, a similar concept will be introduced by means of the following definition.

If x_k is a zero of $r(x)$, q_k will be called its multiplicity bound provided that for every $q > q_k$ there exists a positive constant δ_q such that the inequality

$$|r(x)| \geq |x - x_k|^q \tag{8}$$

holds for all values of x for which $|x - x_k| < \delta_q$, while if $q < q_k$ no such δ_q can be found. It can readily be shown that there exists a unique q_k, finite or infinite, for every zero of a continuous function.

THEOREM 2. *For $0 < m < 1$ the number of zeros and $(1-m)$ times the sum of the zero multiplicity bounds of $r(x)$ must add up to at least n.*

Assume the theorem to be false. Let t represent the number of zeros and u the sum of the zero multiplicity bounds. Then $t + (1-m)u < n$. Let $x_1, x_2, \cdots, x_t$ denote the zeros and $q_1, q_2, \cdots, q_t$ the corresponding multiplicity bounds. It is possible then to construct a polynomial $p(x)$ of proper degree which has the same sign as $r(x)$ everywhere in (a, b), with $|p(x)| \leq 1$ in (a, b), and with zeros of multiplicity p_k satisfying the relation

$$q_k(1-m) - 1 < p_k \leq q_k(1-m) + 1. \tag{9}$$

More explicitly $p(x)$ is expressed by

$$p(x) = C(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_t)^{p_t},$$

where C is a constant. The range of two units in the inequality (9) makes it possible to choose p_k odd or even according as $r(x)$ does or does not change sign at $x = x_k$. From (9) it follows that

$$\sum_{k=1}^{t} p_k \leq \sum_{k=1}^{t} [q_k(1-m) + 1] \leq (1-m)u + t < n. \tag{10}$$

This inequality shows that $p(x)$ is of degree $\leq n-1$. Now form

$$J(h) = \int_a^b w\,|r - hp|^m\,dx,$$

$$J(h) - J(0) = \int_a^b w[|r - hp|^m - |r|^m]dx, \tag{11}$$

where h is a positive constant to be specified later. Let E_1, E_2, and E_3 be the subsets of (a, b) for which respectively $|r| > 2h|p|, h|p|/2 \leq |r| \leq 2h|p|$, and $|r| < h|p|/2$. By construction, $p(x)$ has the same sign as $r(x)$; consequently integral (11) is non-positive over the sets E_1 and E_2 and non-negative over E_3. Moreover, since $J(0)$ is the minimum value of (1), integral (11) must be non-negative for every h. Hence

$$\int_{E_3} w[|r-hp|^m - |r|^m]dx \geq \int_{E_1+E_2} w[|r|^m - |r-hp|^m]dx.$$

A contradiction will be obtained by showing that there exists a value of h for which this inequality is reversed. It will suffice to prove that

$$(12)\quad \int_{E_1} w[|r|^m - |r-hp|^m]dx > \int_{E_3} w[|r-hp|^m - |r|^m]dx.$$

The mean value theorem may be applied to the integrand on the left to give

$$\begin{aligned} |r|^m - |r-hp|^m &= mh|p|[|r| - \theta_2 h|p|]^{m-1} \qquad (0<\theta_2<1), \\ &= mh|p|\,|r|^{m-1}[1-\theta_2 h|p|/|r|]^{m-1} > mh|p|\,|r|^{m-1}. \end{aligned}$$

Hence,

$$\int_{E_1} w[|r|^m - |r-hp|^m]dx \geq mh\int_{E_1} w|p|\,|r|^{m-1}\,dx.$$

Consider the integral on the right of (12). Since $|r| < h|p|/2$ on E_3,

$$\int_{E_3} w[|r-hp|^m - |r|^m]dx \leq \int_{E_3} w|hp|^m\,dx.$$

By combining these two results, it is seen that (12) is true if

$$(13)\qquad m\int_{E_1} w|p|\,|r|^{m-1}\,dx > \int_{E_3} w|p|\,|hp|^{m-1}\,dx.$$

Let h be given a sufficiently small positive value h_0 such that the set E_1 will contain a subset of positive measure on which w is positive. The function $|p|\,|r|^{m-1}$ has but a finite number of zeros; consequently the integral on the left of (13) is positive with h so chosen. Let its value be denoted by K/m. Since the measure of E_1 is non-decreasing as h decreases, this integral is non-decreasing as h decreases. That means, for $h \leq h_0$,

$$(14)\qquad m\int_{E_1} w|p|\,|r|^{m-1}\,dx \geq K(h_0) \equiv K > 0.$$

Over the set E_3, $|hp| > 2|r|$; hence

$$|p|\,|hp|^{m-1} < |p|\,|2r|^{m-1} < |p|\,|r|^{m-1},$$

and

$$\int_{E_3} w\,|\,p\,|\;|\,hp\,|^{m-1}\,dx \leq \int_{E_3} w\,|\,p\,|\;|\,r\,|^{m-1}\,dx. \tag{15}$$

The integrand on the right becomes infinite only at the zeros of $r(x)$. Therefore to prove that the integral exists, it will suffice to show that it converges at any such zero x_k in E_3. The demonstration follows.

It is possible from (9) to find a positive number $\rho < m$ such that $p_k > q_k(1-m) - 1 + \rho$. If $\epsilon > 0$ is chosen so small that $\epsilon(1-m) < \rho$, then

$$p_k > (q_k + \epsilon)(1-m) - 1. \tag{16}$$

From definition (8) there exists a $\delta_\epsilon < 1$ such that $|\,r(x)\,| \geq |\,x - x_k\,|^{q_k+\epsilon}$ for $|\,x - x_k\,| < \delta_\epsilon$. Moreover, from the definition of $p(x)$, there must exist a positive constant R such that $|\,p(x)\,| \leq R\,|\,x - x_k\,|^{p_k}$ for $|\,x - x_k\,| < \delta_\epsilon$. Combination of these last inequalities gives

$$|\,p\,|\;|\,r\,|^{m-1} \leq R\,|\,x - x_k\,|^{(q_k+\epsilon)(m-1)+p_k} \text{ for } |\,x - x_k\,| < \delta_\epsilon. \tag{17}$$

From (16) it follows that the exponent in (17) is > -1. Consequently the integral on the right of (15) converges at x_k, and hence throughout E_3. In view of (14) and (15), it is evident that (13) is true if

$$K > \int_{E_3} w\,|\,p\,|\;|\,r\,|^{m-1}\,dx.$$

If now h is allowed to approach zero, the measure of E_3 approaches zero; and therefore this last integral approaches zero with h. But K is independent of h; consequently an h can be selected so small that the inequality holds. This proves the theorem.

COROLLARY 1. *If the number of zeros is not more than nm, the sum of the zero multiplicity bounds must be at least n.*

THEOREM 3. *If every $q_k < 1/(1-m)$, the number of sign changes of $r(x)$ in (a, b) is at least n.*

Proof. Here $q_k(1-m) + 1 < 2$ and $q_k(1-m) - 1 < 0$; hence, from (9), $p_k = 1$, if $r(x)$ changes sign at x_k, and $p_k = 0$, if it does not. If the theorem is assumed to be false, (10) becomes $\Sigma p_k \leq n - 1$, and the proof follows from there on as in Theorem 2. Under this hypothesis, there can be only a finite number of roots since at a limit point of roots q_k is infinite.

THEOREM 4. *The conclusions of Theorems 2 and 3 and Corollary 1 hold if $w(x)$ is not bounded but is such that $w(x)\,|\,r(x)\,|^{m-1}$ is summable on (a, b).*

Proof. The details of the proof are identical with those of the above theorems through (15), after which the conclusion follows readily from the added summability hypothesis.

3. *Continuity of coefficients.* The problem of determining the behavior of a minimizing sum as the exponent m becomes infinite has been investigated in several papers.[7] The main result of these investigations has been to show that in the case of polynomial approximation to a continuous function the minimizing polynomial, for $m \to \infty$, approaches the Tchebycheff polynomial of best approximation as a limit. However, no attention seems to have been given to the corresponding problem of determining the behavior of a minimizing sum as the exponent m approaches a finite value. The purpose of this section is to investigate that problem to the extent of showing that under suitable restrictions the coefficients of a minimizing sum are continuous functions of the exponent m.

In addition to the original set of restrictions, let $w(x)$ be non-vanishing almost everywhere on (a, b). It will be assumed that $|f(x)| \leq 1$, since if the condition is not fulfilled originally, the said coefficients merely need to be multiplied by a suitable constant factor. It will also be assumed that $f(x)$ and the s's form a set of $n+1$ properly independent functions; for, otherwise, the minimizing sum is identical with $f(x)$ almost everywhere on (a, b), for every m, and the problem of dependence on m becomes trivial. First, a necessary lemma will be proved.

LEMMA. *The absolute values of the coefficients of $\phi_m(x)$ have an upper bound independent of m for $\alpha \leq m \leq \beta$,* where α and $\beta > \alpha$ represent any two positive numbers.

Proof. Let J represent the integral (1) and $\bar{J} = \int_a^b w(x) \, |\theta(x)|^m \, dx$, where $\theta(x)$ is an arbitrary linear combination of $f(x)$ and the s's with its least upper bound $= 1$. Then, by the same proof as that given in a paper by D. Jackson,[8] it can be shown that

$$|c_k| \leq B(J/A_m)^{1/m}, \tag{18}$$

where B is a constant depending only on $f(x)$ and the s's, and A_m is the minimum value of $\bar{J}$. When $J \leq A_m$, $|c_k| \leq B$; otherwise, $|c_k| \leq B(J/A_m)^{1/\alpha}$; and so in all cases

[7] G. Pólya, "Sur un algorithme . . . ," *Comptes Rendus*, vol. 157 (1913), pp. 840-843; D. Jackson, *loc. cit.*, "On functions of closest approximation"; J. Shohat, "On the polynomial of the best approximation to a given continuous function," *Bulletin of the American Mathematical Society*, vol. 31 (1925), pp. 509-514.

[8] *Loc. cit.*, "A generalized problem in weighted approximation."

$$|c_k| \leq B[1 + (J/A_m)^{1/\alpha}]. \tag{19}$$

For a fixed θ, since $|\theta| \leq 1$, $|\theta|^m$ is non-increasing as m increases, for each x, and so $\int_a^b w|\theta|^m dx$ is a non-increasing function of m. Hence, for every θ, $\int_a^b w|\theta|^m dx \geq \int_a^b w|\theta|^\beta dx \geq A_\beta$. Therefore $A_m \geq A_\beta$.

When only minimizing sums are considered,

$$J = \int_a^b w|f - \phi_m|^m dx \leq \int_a^b w|f|^m dx \leq \int_a^b w\, dx,$$

since $|f(x)| \leq 1$ and zero may be regarded as a particular linear combination of the s's.

When these inequalities are applied to (19), $|c_k|$ for $\phi_m(x)$ will be found to have an upper bound independent of m over the range (α, β).

As a consequence of this lemma and the fact that $f(x)$ and the s's are bounded, there must exist a $K > 1$ independent of m such that

$$|f(x) - \phi_m(x)| < K \qquad \text{for } \alpha \leq m \leq \beta. \tag{20}$$

THEOREM 5. *The coefficients of $\phi_m(x)$ are continuous functions of the exponent m for every value of $m > 0$.*

Proof. When $m > 1$, $\phi_m(x)$ is uniquely determined for each value of m. When $m \leq 1$, in the absence of further restrictions, this is not necessarily the case. The property of continuity then requires special explanation, which will be given later. Introduce the notations:

$$I(m, c^m) = \int_a^b w(x)\,|f(x) - \phi_m(x)|^m dx \equiv \int_a^b w|r|^m dx$$

$$I(\mu, c^m) = \int_a^b w(x)\,|f(x) - \phi_m(x)|^\mu dx \equiv \int_a^b w|r|^\mu dx$$

$$I(\mu, c^\mu) = \int_a^b w(x)\,|f(x) - \phi_\mu(x)|^\mu dx \equiv \int_a^b w|z|^\mu dx$$

$$I(m, c^\mu) = \int_a^b w(x)\,|f(x) - \phi_\mu(x)|^m dx \equiv \int_a^b w|z|^m dx.$$

Since $I(m, c^m)$ and $I(\mu, c^\mu)$ are minimized values for exponents m and μ respectively,

$$I(m, c^m) \leq I(m, c^\mu); \qquad I(\mu, c^\mu) \leq I(\mu, c^m). \tag{21}$$

Consider the difference

$$|I(\mu, c^\mu) - I(m, c^\mu)| = \left|\int_a^b w[|z|^\mu - |z|^m]dx\right|.$$

This difference does not exceed a quantity which may be written as

$$(22)\qquad \int_a^b w\,|\,z\,|^m\,|\,|\,z\,|^{\mu-m}-1\,|\,dx, \quad \text{or} \quad \int_a^b w\,|\,z\,|^\mu\,|\,|\,z\,|^{m-\mu}-1\,|\,dx,$$

according as $\mu > m$ or $\mu < m$. Both of these integrals have the limit zero as μ approaches m. This is shown as follows.

Let ϵ be any positive quantity such that $\epsilon_0 = (\epsilon/4W)^{2/\alpha} < 1$, where $W = \int_a^b w(x)\,dx$. Let E_1, E_2, and E_3 be the subsets of (a, b) for which respectively $|\,z\,| \leq \epsilon_0$, $\epsilon_0 < |\,z\,| \leq 1$, and $|\,z\,| > 1$. Choose $|\,\mu - m\,| < \alpha/2$ and consider for $\mu < m$ the last integral in (22) in the three parts corresponding to these three sets. For the first part

$$\int_{E_1} w\,|\,z\,|^\mu\,|\,|\,z\,|^{m-\mu}-1\,|\,dx \leq \int_{E_1} w\,|\,z\,|^\mu\,dx$$
$$\leq \int_{E_1} w\,|\,z\,|^{\alpha/2}\,dx \leq \epsilon_0{}^{\alpha/2}\int_{E_1} w\,dx \leq \epsilon/4.$$

For the second and third parts, because of (20),

$$\int_{E_2+E_3} w\,|\,z\,|^\mu\,|\,|\,z\,|^{m-\mu}-1\,|\,dx \leq K^\beta \int_{E_2+E_3} w\,|\,|\,z\,|^{m-\mu}-1\,|\,dx.$$

The two parts of the integral on the right satisfy the inequalities

$$\int_{E_2} w\,|\,|\,z\,|^{m-\mu}-1\,|\,dx = \int_{E_2} w[1-|\,z\,|^{m-\mu}]dx \leq (1-\epsilon_0{}^{|\mu-m|})\,W.$$
$$\int_{E_3} w\,|\,|\,z\,|^{m-\mu}-1\,|\,dx = \int_{E_3} w[\,|\,z\,|^{m-\mu}-1]dx \leq (K^{|\mu-m|}-1)\,W.$$

Identical results hold for the first integral in (22). Combination of these results gives

$$|\,I(\mu, c^\mu) - I(m, c^\mu)\,| \leq \epsilon/4 + WK^\beta[(1-\epsilon_0{}^{|\mu-m|}) + (K^{|\mu-m|}-1)].$$

Since ϵ_0 and K are independent of $|\,\mu - m\,|$, the second term on the right can be made $\leq \epsilon/4$ by a sufficiently small choice of $|\,\mu - m\,|$, say δ. Hence,

$$(23)\qquad |\,I(\mu, c^\mu) - I(m, c^\mu)\,| \leq \epsilon/2 \qquad \text{for } |\,\mu - m\,| < \delta.$$

Since m and μ may take on any positive values, they are interchangeable here; hence,

$$(24)\qquad |\,I(m, c^m) - I(\mu, c^m)\,| \leq \epsilon/2 \qquad \text{for } |\,\mu - m\,| < \delta.$$

When μ differs from m by less than δ, (21) and (24), and (23) and (21)

combine to give the inequalities $I(\mu, c^\mu) \leq I(\mu, c^m) \leq I(m, c^m) + \epsilon/2$ and $I(\mu, c^\mu) \geq I(m, c^\mu) - \epsilon/2 \geq I(m, c^m) - \epsilon/2$. These are equivalent to

$$|I(\mu, c^\mu) - I(m, c^m)| \leq \epsilon/2 \qquad \text{for } |\mu - m| < \delta. \tag{25}$$

Finally, combining (23) and (25) gives: $|I(m, c^m) - I(m, c^\mu)| \leq \epsilon$ for $|\mu - m| < \delta$. But this says that

$$\lim_{\mu \to m} I(m, c^\mu) = I(m, c^m). \tag{26}$$

The continuity proof results in interpreting this limit as follows. Let μ_i $(i = 1, 2, \cdots)$ be an arbitrary set of values of μ such that $\lim_{i \to \infty} \mu_i = m$. Then from (26)

$$\lim_{i \to \infty} I(m, c^{\mu_i}) = I(m, c^m). \tag{27}$$

When $m > 1$, there corresponds to each value of μ_i a unique set of coefficients c^{μ_i}, which may be thought of as the coördinates of a point in an n-dimensional space. From the lemma, these points are bounded and thus give rise to at least one limit point as i becomes infinite. Since $I(m, c^\mu)$ is a continuous function of c^μ, the limiting value of $I(m, c^{\mu_i})$, as i becomes infinite, is the value taken on for the set corresponding to a limit point. But uniqueness for $m > 1$, because of (27), requires that the set of coefficients corresponding to a limit point be identical with c^m. Hence, there can be but *one* limit point, and $\lim_{i \to \infty} c^{\mu_i} = c^m$. Since μ_i is an arbitrary set approaching m, this is sufficient to prove continuity because it implies that

$$\lim_{\mu \to m} c^\mu = c^m. \tag{28}$$

When $m \leq 1$, the functions $I(m, c^\mu)$ and c^μ are to be thought of as multiple valued functions of μ possessing as many values as there are minimizing sums for that exponent. The reader will find, upon carefully examining the details of the last paragraph, that the same type of argument applies to arrive at (28), the limit to exist in this multiple valued sense. Here the continuity referred to in the theorem means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every μ for which $|\mu - m| < \delta$, every ϕ_μ will have coefficients differing from those of some one of the ϕ_m by less than ϵ.

ROSE POLYTECHNIC INSTITUTE.

REGULAR CONVERGENCE AND MONOTONE TRANSFORMATIONS.

By G. T. WHYBURN.

In a recent paper [1] the author defined the concept of regular convergence for a sequence of sets relative to r-dimensional cycles. A convergent sequence of closed sets $[A_n]$ is said to converge *regularly relative to r-dimensional cycles,* or simply *r-regularly,* provided that for every $\epsilon > 0$ there exist positive numbers δ and N such that if $n > N$ then any r-dimensional complete (Vietoris) cycle in A_n of diameter $< \delta$ is ~ 0 in a subset of A_n of diameter $< \epsilon$. It was shown that in the case of every type of set there studied (i. e., simple arcs, simple closed curves, topological spheres and closed 2-cells), regular convergence relative to 0-cycles for a sequence $[A_n]$ yields as a limiting set A a set either of the same topological type as the members of the sequence or a set which can be produced by an upper semi-continuous decomposition [2] of such a member into continua. In other words, the limiting set is always of such a nature that it can be represented as the image under a monotone transformation [3] of a member of the sequence. This suggests the possibility of an intimate connection existing between regular convergence and monotone transformations, and in the present paper a theorem giving the exact nature of this relationship will be established (see Theorem 2 below).

We shall assume that all sets considered are imbedded in a compact metric space. The distance between two points x and y is denoted by $\rho(x, y)$, the diameter of a set X by $\delta(X)$ and the ϵ-neighborhood of a set X by $V_\epsilon(X)$, i. e., $V_\epsilon(X)$ is the set of all points y such that for some $x \in X$, $\rho(x, y) < \epsilon$. We use the arrow $\rightarrow$ to indicate convergence rather than to show bounding relationships. Thus $A_n \rightarrow A$ means that lim-sup $A_n = \liminf A_n = A$. All our cycles are non-oriented. A complete (Vietoris) r-dimensional cycle will be denoted by γ^r. A closed set A is said to be locally γ^r-connected [4] provided that

[1] See my paper "On sequences and limiting sets," *Fundamenta Mathematicae,* vol. 25 (1935).

[2] See R. L. Moore, "Foundations of point set theory," *American Mathematical Society Colloquium Publications,* 1932, Chapter V.

[3] See C. B. Morrey, *American Journal of Mathematics,* vol. 57 (1935), pp. 17-50, particularly p. 26.

[4] Compare with Alexandroff, *Annals of Mathematics,* ser. 2, vol. 30 (1928), p. 181; and Lefschetz, "Topology," *American Mathematical Society Colloquium Publications,*

for every $\epsilon > 0$ there exists a $\delta > 0$ such that any γ^r in A of diameter $< \delta$ is ~ 0 in a subset of A of diameter $< \epsilon$. For a systematic treatment of the combinatorial notions used the reader is referred to the papers of Vietoris and Alexandroff.[5]

THEOREM 1. *If the sequence of closed sets $[A_n]$ converges to the limiting set A regularly relative to s-cycles for every $s \leqq r$, then A is locally γ^r-connected.*

Proof. By virtue of a previous theorem of the author's [6] we have only to show that for $e > 0$ there exists a $d > 0$ such that, for any $\epsilon > 0$, positive numbers δ and N exist such that if $n > N$, then any r-dimensional δ-cycle in A_n of diameter $< d$ is $\underset{\epsilon}{\sim} 0$ in a subset of A_n of diameter $< e$. To this end let $e > 0$ be given. By virtue of the r-regular convergence there exist positive numbers d and N_r with $3d < e/3$ such that if $n > N_r$ then any γ^r in A_n of diameter $< 3d$ is ~ 0 in a subset of A_n of diameter $< e/3$.

Now let $\epsilon > 0$ be given and let $\delta_r = \min(\epsilon, d)$. By virtue of the $(r-1)$-regular convergence, there exist positive numbers δ_{r-1} and N_{r-1} such that if $n > N_{r-1}$, then any γ^{r-1} in A_n of diameter $< 3\delta_{r-1}$ is ~ 0 in a subset of A_n of diameter $< \delta_r$. Likewise, by virtue of the $(r-2)$-regular convergence, using δ_{r-1} we have positive numbers δ_{r-2} and N_{r-2} such that if $n > N_{r-2}$, then any γ^{r-2} in A_n of diameter $< 3\delta_{r-2}$ is ~ 0 in a subset of A_n of diameter $< \delta_{r-1}$. Similarly, using δ_{r-2}, the $(r-3)$-regular convergence gives us numbers δ_{r-3} and N_{r-3}, and so on. Continuing in this way we reach numbers δ_0 and N_0 such that if $n > N_0$, then any γ^0 in A_n of diameter $< 3\delta_0$ is ~ 0 in a subset of A_n of diameter $< \delta_1$. We may suppose the δ's chosen so that $\delta_r > \delta_{r-1} > \cdots > \delta_0$. Let us now take $\delta = \delta_0$ and $N = \sum_0^r N_i$. We shall show that this choice of δ and N satisfies the required conditions for the given ϵ relative to d and e.

To this end let n be any integer $> N$ and let C^r be any r-dimensional δ-cycle in A_n of diameter $< d$. Let $\Delta = (x_0, x_1, \cdots, x_r)$ be any r-simplex in C^r. Now since $\delta(\Delta) < \delta_0$, it follows by 0-regular convergence that each 1-dimensional side (x_i, x_j) of Δ bounds a 1-dimensional semi-cycle [7] x_{ij} in A_n

1930, pp. 91-92. See also the author's paper in the *American Journal of Mathematics*, vol. 56 (1934), pp. 133-146.

[5] See, for example, Vietoris, *Mathematische Annalen*, vol. 97 (1927), pp. 454-472, and *Fundamenta Mathematicae*, vol. 19 (1932), pp. 265-273; Alexandroff, *loc. cit.*, pp. 101-187, and *Mathematische Annalen*, vol. 106 (1932), pp. 161-238.

[6] See my paper "On sequences and limiting sets," *loc. cit.*, Theorem (1.4).

[7] If $s > 0$ and $Z^{s-1} = (z_1, z_2, \cdots)$ is a complete cycle of dimension $s-1$, then by an s-dimensional *semi-cycle* bounded by Z^{s-1} is meant a sequence of s-dimensional complexes $C^s = (c_1, c_2, \cdots)$ where c_i is a σ_i-complex bounded by z_i, where $\sigma_i \to 0$ and where, for any given $e > 0$, $c_i \underset{e}{\sim} c_{i+j}$, for i sufficiently large, in the sense that if K_i is

of diameter $< \delta_1$. Now for each 2-dimensional side (x_i, x_j, x_k) of Δ, $c_{ijk} = x_{ij} + x_{ik} + x_{jk}$ is a γ^1 in A_n of diameter $< 3\delta_1$. Hence by 1-regular convergence, since $N > N_1$, it follows that c_{ijk} bounds a 2-dimensional semi-cycle x_{ijk} in A_n of diameter $< \delta_2$. Similarly, for each 3-dimensional side (x_i, x_j, x_k, x_m) of Δ, $c_{ijkm} = x_{ijk} + x_{ijm} + x_{ikm} + x_{jkm}$ is a γ^2 in A_n of diameter $< 3\delta_2$. Thus c_{ijkm} bounds a 3-dimensional semi-cycle x_{ijkm} in A_n of diameter $< \delta_3$. Continuing in this manner we eventually obtain, for each $(r-1)$-dimensional side $(x_{i_0}, x_{i_1}, \cdots, x_{i_{r-1}})$ of Δ a γ^{r-2}, $c_{i_0 i_1 \cdots i_{r-1}}$ in A_n of diameter $< 3\delta_{r-2}$. By virtue of the $(r-2)$-regular convergence, since $N > N_{r-2}$, this γ^{r-2} bounds an $(r-1)$-dimensional semi-cycle $x_{i_0 i_1 \cdots i_{r-1}}$ in A_n of diameter $< \delta_{r-1}$. Then $c_{12\cdots r} = \Sigma x_{i_0 i_1 \cdots i_{r-1}}$ is a γ^{r-1} in A_n of diameter $< 3\delta_{r-1}$. Accordingly it bounds an r-dimensional semi-cycle $x_{12\cdots r}$ in A_n of diameter $< \delta_r$. The sum of all such semi-cycles $x_{12\cdots r}$ for all r-simplexes Δ in C^r is a γ^r, say Z, in A_n of diameter $< d + 2\delta_r \leq 3d$. Furthermore, if we write $Z = (z_1, z_2, \cdots)$ as a γ^r, it is readily seen that we have $C^r \underset{\delta_r}{\sim} z_k$ in a subset of A_n of diameter $< d + 2\delta_r \leq 3d < e/3$, for every k. Whence, $C^r \underset{\epsilon}{\sim} z_k$ for all k, since $\delta_r \leq \epsilon$.

Now since $\delta(Z) < 3d$ and since $n > N > N_r$, it follows by r-regular convergence that $Z \sim 0$ in a subset of A_n of diameter $< e/3$. Thus for k sufficiently large, we have $z_k \underset{\epsilon}{\sim} 0$ in a subset of A_n of diameter $< e/3$. Therefore we have $C^r \underset{\epsilon}{\sim} z_k \underset{\epsilon}{\sim} 0$ in a subset of A_n of diameter $< e$, as was to be proven.

COROLLARY. *Under the hypothesis of Theorem* 1, *A is locally γ^s-connected for every $s \leq r$.*

Definition. If A is compact and metric and the transformation $T(A) = B$ is continuous, then T is said to be *r-monotone* provided that for each $b \in B$ all the connectivity (Betti) numbers of $T^{-1}(b)$ of dimension $\leq r$ vanish. For the case $r = 0$ this reduces to the ordinary monotone transformation as defined by Morrey (*loc. cit.*).

THEOREM 2. *Let the sequence of r-monotone transformations $T_n(A) = B_n$ converge uniformly to the limit transformation $T(A) = B$. In order that the sequence $[B_n]$ converge to B s-regularly for every $s \leq r$ it is necessary and sufficient that T be r-monotone and that B be locally γ^s-connected for every $s \leq r$.*

Proof. To prove the conditions necessary, suppose that $B_n \to B$ s-regularly

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any s-dimensional complex of sufficiently small norm bounded by $z_i + z_{i+j}$, the s-cycle $K_i + c_i + c_{i+j}$ bounds an $(s+1)$-dimensional ϵ-complex.

for every $s \leq r$. Let s be any integer $\leq r$, let $b \in B$, let $K = T^{-1}(b)$ and let $\epsilon > 0$ be given. We shall show first that every γ^s in K is ~ 0 in $V_\epsilon(K)$.

Now there exists a neighborhood R of b such that $T^{-1}(B \cdot \bar{R}) \subset V_{\epsilon/2}(K)$. There exists an integer N_1 such that for every $n > N_1$, $T_n^{-1}(B_n \cdot \bar{R}) \subset V_\epsilon(K)$. Also, by virtue of the s-regular convergence, there exists an N_2 such that for every $n > N_2$, $T_n(K) \subset R$ and every γ^s in $T_n(K)$ is ~ 0 in $B_n \cdot R$. Now let C be any γ^s in K. Let n be a fixed integer $> N_1 + N_2$. Then $T_n(C)$ is a γ^s in $B_n \cdot \bar{R}$; and since $T_n(C) \subset T_n(K)$, we have $T_n(C) \sim 0$ in $B_n \cdot \bar{R}$. Now, taking $B_n \cdot \bar{R} = \mathfrak{R}$ and $T_n^{-1}(B_n \cdot \bar{R}) = \mathbf{K} \subset V_\epsilon(K)$ and applying Vietoris' result (10),[8] it follows that since $T_n(C) \sim 0$ in $B_n \cdot \bar{R}$, C must be ~ 0 in $T_n^{-1}(B_n \cdot \bar{R})$. Whence, $C \sim 0$ in $V_\epsilon(K)$.

Now from this it results at once that any such cycle C in K must be ~ 0 in K. For let $C = (z_1, z_2, \cdots)$ and let $\epsilon > 0$ be given. By what was just shown it follows that for k sufficiently large, z_k bounds an $\epsilon/3$-complex K_0^{s+1} in $V_{\epsilon/3}(K)$; and clearly this complex projects by an $\epsilon/3$-projection [9] into an ϵ-complex K^{s+1} in K bounded by z_k. Thus we have shown that any complete cycle C in K of dimension $\leq r$ is ~ 0 in K, and accordingly T is r-monotone.

Finally, that B is locally γ^s-connected for every $s \leq r$ is a consequence of Theorem 1. Hence the conditions are necessary.

To prove the sufficiency of the conditions, we suppose the contrary. It follows that for some $s \leq r$, some $\epsilon > 0$ and some infinite sequence of integers (m_n) we have carriers $C_n \subset B_{m_n}$ of complete s-dimensional cycles Z_n such that $C_n \to b \in B$ but such that for no n is $Z_n \sim 0$ in any subset of $B_{m_n} \cdot V_{2\epsilon}(b)$.

We note first that if $W_n(x) = T_n T^{-1}(x)$, for each $x \in B$, then $[W_n(x)]$ converges uniformly to the identity transformation $W(x) = x$ on B. Now for the given ϵ, let us determine $\delta < \epsilon$ as given by the local γ^s-connectivity of B. Then by the uniform convergence $W_n(x) \to W(x)$, it follows that there exists an N such that if $n > N$ then $W_n(x) \subset V_{\delta/2}(x)$ for every $x \in B$. This gives

$$\text{(i)} \qquad W_n[B \cdot V_{\delta/2}(b)] \subset V_\delta(b) \quad \text{and} \quad W_n^{-1}[B_n \cdot V_{\delta/2}(b)] \subset V_\delta(b);$$

and since $\epsilon > \delta$, we have likewise

$$\text{(ii)} \qquad W_n[B \cdot V_\epsilon(b)] \subset V_{2\epsilon}(b).$$

Now let us determine an integer k so that $m_k > N$ and $C_k \subset V_{\delta/2}(b)$. Then since $C_k \subset B_{m_k}$, (i) gives $W_{m_k}^{-1}(C_k) \subset V_\delta(b)$. Now by Vietoris' result (6)

[8] See *Mathematische Annalen*, vol. 97 (1927), p. 469.

[9] That is, for each vertex x_0 in K_0^{s+1} which is not already in K, we substitute a point x of K with $\rho(x, x_0) < \epsilon/3$.

[*loc. cit.*, using $\mathfrak{N} = C_k$ and $\mathbf{K} = T_{m_k}^{-1}(C_k)$], it follows that $T_{m_k}^{-1}(C_k)$ carries an s-dimensional complete cycle Z'_k such that $T_{m_k}(Z'_k) \sim Z_k$ in C_k. Since $T(Z'_k) \subset V_\delta(b)$, it follows that $T(Z'_k) \sim 0$ in $B \cdot V_\epsilon(b)$. Thus, since T is r-monotone, it follows by Vietoris' result (10) (*loc. cit.*) that $Z'_k \sim 0$ in $T^{-1}[B \cdot V_\epsilon(b)]$; and by (ii) and the continuity of T_{m_k} it follows that $T_{m_k}(Z'_k) \sim 0$ in $B_{m_k} \cdot V_{2\epsilon}(b)$. But since $Z_k \sim T_{m_k}(Z'_k)$ in C_k and $C_k \subset B_{m_k} \cdot V_\delta(b)$, this gives $Z_k \sim 0$ in $B_{m_k} \cdot V_{2\epsilon}(b)$, contrary to supposition. Thus the supposition that the conditions are not sufficient leads to a contradiction, and the theorem is proven.

COROLLARY. *If the sequence of r-monotone transformations $T_n(A) = B$ converges uniformly to the limit transformation $T(A) = B$ and if B is locally γ^s-connected for every $s \leq r$, then T is r-monotone.*

For, in the case of a sequence of the type $[B, B, B, \cdots]$, it is readily seen that s-regular convergence of the sequence and local γ^s-connectivity of the set B are equivalent.

Note. Let $r = 0$ and let us suppose all the transformations T_n in Theorem 2 are homeomorphisms. Consider the following choices for A: (1) the interval $(0, 1)$, (2) the circle $x^2 + y^2 = 1$, (3) the sphere $x^2 + y^2 + z^2 = 1$, and (4) the closed circular region $x^2 + y^2 \leqq 1$. Then the sequence $[B_n]$ is, respectively, a sequence of (1) simple arcs, (2) simple closed curves, (3) topological spheres and (4) closed 2-cells. Now Theorem 2 tell us that if $B_n \to B$ 0-regularly, then the transformation $T(A) = B$ is monotone ($= 0$-monotone); and thus by known results,[10] in these cases, B is respectively, a (a) simple arc, (b) simple closed curve, (c) cactoid, (d) hemi-cactoid. Thus it might appear at first glance that we have obtained the principal end-results of the author's paper "On sequences and limiting sets" (*loc. cit.*) by direct application of the more general Theorem 2. However, this is far from the case. For, in order to obtain these conclusions about B in the cases (1)-(4), the "analytic" or "transformation" representations $T_n(A) = B_n$ of the sets B_n on a base set A are entirely *superfluous*. It may very well be possible, given a 0-regularly convergent sequence $[B_n]$ of sets of types (1)-(4), to introduce the transformation representations in such a way as to get uniform convergence; but until this can be demonstrated, the direct methods employed in the earlier paper, even though restricted for the present to the lower dimensions, are much more effective in treating cases such as the ones here exemplified.

UNIVERSITY OF VIRGINIA.

[10] See Moore, *loc. cit.*, and Morrey, *loc. cit.*

ON SOME ASYMPTOTIC RELATIONS FOR THE CHARACTERISTIC VALUES OF THE ELLIPTIC DIFFERENTIAL EQUATIONS.

By H. L. KRALL.

Let

$$L(u) = A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u$$

be an elliptic differential expression defined in the region T which is bounded by a simple closed curve R. We assume that R has a continuously turning tangent and that the coefficients $A(x,y), B(x,y), \cdots, F(x,y)$ have continuous partial derivatives of the first three orders in $T+R$.

Let $\lambda_1, \lambda_2, \cdots$ $(\lambda_i \leqq \lambda_{i+1})$ be the C. V. (characteristic values) of the differential system

$$\text{(A)} \qquad Lu = \lambda v, \qquad Mv = \lambda u$$
$$[u]_{(x,y)\text{ on }R} = [v]_{(x,y)\text{ on }R} = 0,$$

where M is the operator adjoint to L and each C. V. is counted a number of times equal to its multiplicity.

In this paper we propose to show that [1]

$$\lambda_n \cong \frac{4\pi n}{\iint_T \frac{dxdy}{\sqrt{AC - B^2}}}.$$

This result was proved by Geppert [2] [3] for the case $A = C = 1$, $B = 0$, and then from this result of Geppert's it was shown by Gheorghiu [4] that if $\sigma_1, \sigma_2, \cdots$ $(|\sigma_i| \leqq |\sigma_{i+1}|)$ are the C. V. of the system

$$\text{(B)} \qquad Lz + \sigma z = 0$$
$$[z]_R = 0,$$

the series

$$\sum_{n=1}^{\infty} \frac{1}{|\sigma_n|^{1+\epsilon}}$$

converges for every $\epsilon > 0$. We shall show that this result is also true for the general case.

[1] $\cong$ is read "is asymptotically equal to."

[2] See the bibliography at the end of the paper. The number in square brackets indicates the particular article referred to.

Discussion of the system (A). By a solution of (A) we shall mean a pair of functions satisfying (A) and possessing continuous partial derivatives of the first two orders in $T+R$. These functions also satisfy the integral equations

$$(C)\qquad \begin{aligned} u(x,y) &= \lambda \iint_T \Gamma(x,y,\xi,\eta)v(\xi,\eta)\,d\xi d\eta \\ v(x,y) &= \lambda \iint_T \Gamma(\xi,\eta,x,y)u(\xi,\eta)\,d\xi d\eta, \end{aligned}$$

and as well the equations

$$u(x,y) = \lambda^2 \iint_T \Gamma_1(x,y,\xi,\eta)u(\xi,\eta)\,d\xi d\eta;$$

$$\Gamma_1(x,y,\xi,\eta) = \iint_T \Gamma(x,y,r,s)\Gamma(\xi,\eta,r,s)\,drds,$$

$$v(x,y) = \lambda^2 \iint_T \Gamma_2(x,y,\xi,\eta)v(\xi,\eta)\,d\xi d\eta;$$

$$\Gamma_2(x,y,\xi,\eta) = \iint_T \Gamma(r,s,x,y)\Gamma(r,s,\xi,\eta)\,drds,$$

where $\Gamma(x,y,\xi,\eta)$ is the Green's function (or extended Green's function) of the operator L with the boundary condition $[\Gamma(x,y,\xi,\eta)]_{(x,y)\text{ on }R} = 0$. The existence of such a function has been proven by Girard ([5] see especially page 222).

E. Schmidt [8] has shown that the C. V. of (C) are positive and that the corresponding solutions build a complete set. Hence any piecewise continuous function can be approximated in the mean by a Fourier series of the $\{u_n\}$ or $\{v_n\}$ where u_n, v_n are normalized, $\left(\iint_T u_n^2\,dxdy = 1\right)$, characteristic functions corresponding to λ_n.

THEOREM. *The* C. V. *of* (A) *are identical with the* C. V. *of*

$$(D)\qquad \begin{aligned} MLu - \lambda^2 u &= 0 \\ [u]_R = [Lu]_R &= 0, \end{aligned}$$

and

$$(D')\qquad \begin{aligned} LMv - \lambda^2 v &= 0 \\ [v]_R = [Mv]_R &= 0. \end{aligned}$$

Moreover the multiplicity of each C. V. *is the same.*

By a solution of (D) or (D′) we shall mean a function satisfying the system and possessing continuous partial derivatives of the first three orders and piecewise continuous derivatives of the fourth order.

The solutions of (A) have continuous derivatives of the first four orders (Hopf [6], p. 210); accordingly, we may operate with M and L respectively, which shows that the solutions of (A) satisfy (D) and (D′).

Now suppose that λ' is a C. V. of (D) but not of (A). Then

$$MLu' - \lambda'^2 u' = 0$$
$$MLu_n - \lambda_n^2 u_n = 0.$$

Multiply the first by u_n, the second by u', subtract and integrate, then

$$0 = \iint_T [u_n MLu' - u' MLu_n]\,dxdy - (\lambda_n^2 - \lambda'^2) \iint_T u_n u'\,dxdy,$$

$$\iint_T u_n u'\,dxdy = 0.$$

Since the set $\{u_n\}$ is complete, it follows that $u' \equiv 0$ and λ' is not a C. V.

Suppose now that associated with the C. V. λ, there are n linear independent normalized $\left(\iint u_n^2\,dxdy = 1\right)$ solutions of (D) namely $u_1, u_2, \cdots, u_n$, and t $(t \leqq n)$ linear independent solutions of (D′), $v_1, v_2, \cdots, v_t$. We shall produce n linear independent solutions of (A) which will show that the multiplicity is the same throughout.

Let

$$\bar{u}_s = u_s, \qquad \bar{v}_s = \sum_{j=1}^{t} a_{sj} v_j, \qquad a_{sj} = (1/\lambda) \iint_T v_j L(u_s)\,dxdy.$$

We wish to show that $(\bar{u}_s, \bar{v}_s)$ is a solution of (A). Set

$$f_s = L\bar{u}_s - \lambda \bar{v}_s = Lu_s - \lambda \sum_{j=1}^{t} a_{sj} v_j,$$
$$g_s = M\bar{v}_s - \lambda \bar{u}_s = \sum_{j=1}^{t} a_{sj} M(v_j) - \lambda u_s.$$

Then

$$Mf_s = -\lambda g_s, \qquad LMf_s = \lambda^2 f_s,$$
$$Lg_s = -\lambda f_s, \qquad MLg_s = \lambda^2 g_s.$$

Accordingly $(-g_s, f_s)$ is a solution of (A) and f_s is a solution of (D′) and we must have $f_s = \sum_{j=1}^{t} a_{sj} v_j$. Then

$$a_{sj} = \iint_T f_s v_j\,dxdy = \iint_T v_j Lu_s\,dxdy - \lambda \sum_{k=1}^{t} a_{sk} \iint_T v_j v_k\,dxdy$$
$$= \iint_T v_j Lu_s\,dxdy - \lambda a_{sj} = 0.$$

Since $f_s = 0$ we must have $g_s = 0$ and then $(\bar{u}_s, \bar{v}_s)$ are solutions of (A). Moreover we must also have $n = t$, for otherwise there would exist non-zero constants $\{c_i\}$ such that $\sum_{i=1}^{n} c_i \bar{v}_i = 0$, and then

$$0 = \sum_{i=1}^{n} c_i M\bar{v}_i = \lambda \sum_{i=1}^{n} c_i \bar{u}_i,$$

which is a contradiction.

The maximum-minimum property of the C. V. In the following theorem (Courant-Hilbert [2, p. 326]), we use ϕ to denote a general function having continuous partial derivatives of the first three orders in $T + R$ and piecewise continuous partial derivatives in T.

THEOREM. *Let $\xi_1, \xi_2, \cdots, \xi_{n-1}$ be arbitrary piecewise continuous functions in T, and let $g(\xi_1, \xi_2, \cdots, \xi_{n-1})$ be the lower bound or minimum of*

$$K(\phi) = \int_T \int [L(\phi)]^2 \, d\sigma$$

for all admissible functions ϕ satisfying

$$\int_T \int \phi^2 d\sigma = 1$$

$$\int_T \int \phi \xi_i d\sigma = 0 \qquad (i = 1, 2, \cdots, n-1)$$

$$[\phi]_R = 0.$$

Let λ_n be the n-th C. V. *of* (D). *Then λ_n^2 is the maximum value which $g(\xi_1, \xi_2, \cdots, \xi_{n-1})$ can assume, and this maximum-minimum value will be taken for $\xi_1 = u_1, \xi_2 = u_2, \cdots, \xi_{n-1} = u_{n-1}, \phi = u_n$.*

The Euler equation of this Calculus of Variations problem is $MLu - \lambda^2 u = 0$, and this equation must of course be satisfied by the minimizing function. We would not change the lower bound by demanding that $[L(\phi)]_R = 0$, because (Geppert [3]) we can approximate every ϕ by an admissible function ψ such that $[L(\psi)]_R = 0$ and $K(\phi)$ differs from $K(\psi)$ by an arbitrarily small amount.

To prove the theorem consider

$$w(P) = \sum_{i=1}^{n} c_i u_i(P), \qquad (u_i(P) = u_i(x, y)).$$

The $\{c_i\}$ can be determined so that $w(P)$ is an admissible function. Then using Green's formula we have

$$K(w) = \int\int_T [L(w)]^2 d\sigma = \int\int_T wMLw d\sigma = \sum_{i=1}^{n} c_i^2 \lambda_i^2 \leqq \lambda_n^2 \sum_{i=1}^{n} c_i^2 = \lambda_n^2.$$

Accordingly

$$g(\xi_1, \xi_2, \cdots, \xi_{n-1}) \leqq K(w) \leqq \lambda_n^2.$$

Now take $\xi_1 = u_1, \xi_2 = u_2, \cdots, \xi_{n-1} = u_{n-1}$. The function ϕ generates the series

$$\phi(P) \sim \sum_{i=1}^{\infty} \phi_i u_i(P), \qquad (\phi_i = \int\int_T \phi u_i d\sigma).$$

Making use of Parseval's identity and the identities

$$\int\int_T fg d\sigma = \sum_{n=1}^{\infty} f_i g_i, \quad \int\int_T u_i ML\phi d\sigma = \int\int_T \phi MLu_i d\sigma = \lambda_i^2 \phi_i,$$

we have, since $\phi_1 = \phi_2 = \cdots = \phi_{n-1} = 0$,

$$K(\phi) = \int\int_T \phi ML\phi d\sigma = \sum_{i=1}^{\infty} \lambda_i^2 \phi_i^2 \geqq \lambda_n^2 \sum_{i=1}^{\infty} \phi_i^2 = \lambda_n^2.$$

But for $\phi = u_n$ we have $K(\phi) = \lambda_n^2$, which completes our proof.

Transformation of $K(\phi)$. *Lichtenstein* [7, p. 38] considered the expression

$$L_1(w) = A_1 w_{xx} + 2B_1 w_{xy} + C_1 w_{yy} + D_1 w_x + E_1 w_y + F_1 w$$

where $A_1 C_1 - B_1^2 = 1$. H showed that functions $X(x, y)$, $Y(x, y)$ exist which are solutions of $(\partial/\partial x)(A_1 U_x + B_1 U_y) + (\partial/\partial y)(B_1 U_x + C_1 U_y) = 0$, and which transform the region in a one to one manner into the region T', so that, setting $w(x, y) = W(X, Y)$

$$\begin{aligned} L_1(w) = L'_1(W) &= \delta(W_{XX} + W_{YY}) + D' W_X + E' W_Y + FW \\ \delta &= A_1 X_x^2 + 2B_1 X_x X_y + C_1 X_y^2, \\ D' &= \{D_1 - (A_1)_x - (B_1)_y\} X_x + \{E_1 - (B_1)_x - C_1)_y\} X_y \\ E' &= \{D_1 - (A_1)_x - (B_1)_y\} Y_x + \{E_1 - (B_1)_x - (C_1)_y\} Y_y \\ F' &= F_1. \end{aligned}$$

These functions $X(x, y)$, $Y(x, y)$ must have continuous partial derivatives of the first four orders (Hopf [6], p. 210). The Jacobian of the transformation is δ, and

$$dxdy = (1/\delta) dXdY \qquad (\delta > 0).$$

Setting $\alpha = (AC - B^2)^{1/2}$, $A_1 = A/\alpha$, $B_1 = B/\alpha, \cdots, F_1 = F/\alpha$, we can write

$$K(\phi) = \int\int_T \alpha^2 [L_1(\phi)]^2 \, dxdy.$$

Thus the transformation gives us

$$K(\phi) = K(\Phi) = \int\int_{T'} (\alpha^2/\delta)(\delta\Phi_{XX} + \delta\Phi_{YY} + D'\Phi_X + E'\Phi_Y + F'\Phi)^2 dXdY$$
$$= \int\int_{T'} [\mathfrak{L}(\Phi)]^2 dXdY,$$

where

$$\phi(x, y) = \Phi(X, Y), \qquad \mathfrak{L}(\Phi) = p^{1/2}(\Phi_{XX} + \Phi_{YY}) + a\Phi_X + b\Phi_Y + f\Phi,$$
$$p^{1/2} = \alpha\delta^{1/2}, \qquad b = \alpha E'/\delta^{1/2},$$
$$a = \alpha D'/\delta^{1/2}, \qquad f = \alpha F'/\delta^{1/2}.$$

In the new variables the conditions on ϕ become

$$\int\int_T \phi^2 d\sigma = \int\int_{T'} \rho\Phi^2 dXdY = 1 \qquad (\rho = 1/\delta > 0),$$
$$\int\int_T \phi\xi_i d\sigma = \int\int_{T'} \rho\Phi\xi_i dXdY = 0 \qquad (i = 1, 2, \cdots, n-1),$$
$$[\phi]_R = [\Phi]_{R'} = [L(\phi)]_R = [\mathfrak{L}(\Phi)]_{R'} = 0.$$

Accordingly

$$\lambda_n^2 = \text{Maximum-Minimum } K(\phi)$$

where the admissible functions Φ satisfy the above conditions.

In the same manner as before we can show that the $\{\lambda_n\}$ are the C. V. of the system

$$\text{(E)} \qquad \begin{gathered} \mathfrak{M}\mathfrak{L}U - \lambda^2\rho U = 0 \\ [U]_{R'} = [\mathfrak{L}(U)]_{R'} = 0 \\ \mathfrak{M} = \text{operator adjoint to } \mathfrak{L}. \end{gathered}$$

Our proof of the maximum-minimum property required that the set $\{U_n\}$ of characteristic functions of (E) be complete. This completeness follows from the fact that the system (E) is equivalent to the integral equation

$$U(P) = \lambda^2 \int\int_{T'} \Big[\int\int_{T'} G(P, N)G(Q, N)d\sigma_N\Big] \rho(Q)U(Q)d\sigma_Q$$

where $G(P, Q)$ is the Green's function (ordinary or extended) of the operator L.

An asymptotic relation for the C. V. We start with a discussion of the system

$$\text{(F)} \qquad \begin{matrix} \Delta(p\,\Delta W) - l^2\rho W = 0 \\ [W]_{R'} = [\Delta W]_{R'} = 0 \end{matrix} \qquad (\Delta = \partial^2/\partial X^2 + \partial^2/\partial Y^2)$$

where the p and ρ are defined above, and of course are greater than zero in T'. Then

$$l_n^2 = \text{Max-Min } H(\Phi) = \text{Max-Min} \int\int_{T'} p(\Delta\Phi)^2 dXdY$$

where the admissible functions Φ satisfy the same conditions as before.

Let $A(l)$ denote the number of C. V. less than or equal to l. Now if $p(P)$ is replaced by $p'(P)$ where $p' \geqq p$ the minimums $h(\xi_1, \xi_2, \cdots, \xi_{n-1})$ of $H(\Phi)$ cannot be decreased. Hence $l_n^2 = h(W_1, W_2, \cdots, W_{n-1})$ cannot be decreased and $A(l)$ cannot be increased. If $\rho(P)$ is replaced by $\rho'(P)$ where $\rho' \leqq \rho$ we introduce $\Phi' = k\Phi$ where k is such that

$$1 = \int\int_{T'} \rho'\Phi'^2 d\sigma = k^2 \int\int_{T'} \rho'\Phi^2 d\sigma \leqq k^2 \int\int_{T'} \rho\Phi^2 d\sigma = k^2.$$

Consequently $k \geqq 1$ and

$$H(\Phi') = k^2 H(\Phi) \geqq H(\Phi).$$

Accordingly l_n cannot be decreased and $A(l)$ cannot be increased.

We now use the following lemma (Courant and Hilbert [2], p. 330).

LEMMA. *Let $T_1, T_2, \cdots$ be non-overlapping sub-regions of T and let $A_{T_i}(l)$ designate the number of* C. V. *less than or equal to l for the region T_i, then*

$$A(l) \geqq A_{T_1}(l) + A_{T_2}(l) + \cdots.$$

We divide T' up into a network of squares T_i of side δ. In square T_i replace p by its maximum p_{M_i} and replace ρ by its minimum ρ_{m_i}, then $A_{T_i}(l)$ cannot be increased. Accordingly if $A'_{T_i}(l)$ refers to

$$\begin{matrix} p_{M_i}\Delta\Delta\Phi - l^2\rho_{m_i}\Phi = 0 \\ [\Phi]_R = [\Delta\Phi]_R = 0, \end{matrix}$$

we must have

$$A_{T_i}(l) \geqq A'_{T_i}(l).$$

But this last system is equivalent to the system

$$\begin{matrix} \Delta\Phi - l\sqrt{\dfrac{\rho_{m_i}}{p_{M_i}}}\,\Phi = 0 \\ [\Phi]_R = 0, \end{matrix}$$

and we have (Courant and Hilbert [2], p. 355)

$$A'_{T_i}(l) \cong \frac{\delta^2}{4\pi}\sqrt{\frac{\rho_{m_i}}{p_{M_i}}}\, l.$$

Hence

$$A(l) \gtrsim A'_{T_1}(l) + A'_{T_2}(l) + \cdots$$
$$\gtrsim \frac{l}{4\pi}\sum_i \delta^2 \sqrt{\frac{\rho_{m_i}}{p_{M_i}}} = \frac{l}{4\pi}\left\{ \int_{T'}\int \sqrt{\frac{\rho}{p}}\, d\sigma + \epsilon(\delta) \right\}$$

$$A(l) \gtrsim \frac{l}{4\pi}\int_{T'}\int \sqrt{\frac{\rho}{p}}\, d\sigma; \qquad l_n \gtrsim \frac{4\pi n}{\int_{T'}\int \sqrt{\frac{\rho}{p}}\, d\sigma}.$$

To obtain an asymptotic lower bound for l_n set

$$S = \sqrt{\rho p}, \qquad D(\Phi) = \int_{T'}\int S(\Phi_X^2 + \Phi_Y^2)\, d\sigma$$

then

$$\sqrt{H(\Phi)} = \sqrt{\int\int p(\Delta\Phi)^2 d\sigma \int\int \rho\Phi^2 d\sigma} \gtrsim \left| \int_{T'}\int S\Phi\,\Delta\,\Phi d\sigma \right|$$
$$= \left| D(\Phi) + \int_{T''}\int \Phi(S_X\Phi_X + S_Y\Phi_Y)\, d\sigma \right|$$
$$\sqrt{H(\Phi)} \gtrsim |D(\Phi)| - \left| \int_{T}\int \Phi(S_X\Phi_X + S_Y\Phi_Y)\, d\sigma \right|$$
$$\sqrt{H(\Phi)} \gtrsim D(\Phi) - c\, D(\Phi)^{1/2} \qquad (c = \text{a positive constant}).$$

Let τ_n be the maximum-minimum value of $D(\theta)$ where the admissible functions θ are continuous and have piecewise continuous first derivatives and satisfy

$$\int_{T'}\int \rho\theta^2 d\sigma = 1$$
$$\int_{T'}\int \rho\theta\xi_i d\sigma = 0 \qquad (i = 1, 2, \cdots, n-1)$$
$$[\theta]_R = 0.$$

This $D(\theta)$ can be approximated by $D(\Phi)$ where Φ is some function of our former class. Hence τ_n is the n-th C. V. of the system

$$(S\theta_X)_X + (S\theta_Y)_Y + \tau\rho\theta = 0$$
$$[\theta]_{R'} = 0.$$

We have the relation (Courant and Hilbert [2], p. 355)

$$\tau_n \cong \frac{4\pi n}{\iint_{T'} (\rho/S)\, d\sigma}.$$

Now

$$\sqrt{h(\xi_1, \xi_2, \cdots, \xi_{n-1})} = \text{Lower Bound } \sqrt{H(\Phi)} \geqq \text{Lower Bound } [D(\Phi) - cD(\Phi)^{1/2}].$$

If we draw the graph of the curve $y = x - cx^{1/2}$ we see that the lower bound on the right is either less than zero or is taken on when $D(\Phi)$ is least. Hence if n is so large that $\tau_n > c^2$, we can find a set of $\{\xi_i\}$ so that lower bound of $D(\Phi)$ is greater than c^2 and for this set, the lower bound will be taken on when $D(\Phi)$ is least. Thus for $s \geqq n$ we have

$$\begin{aligned} l_s &= \text{Max-Min } \sqrt{H(\Phi)} \\ &\geqq \text{Minimum } \sqrt{H(\Phi)} \text{ for } \xi_i = \theta_i \qquad (i = 1, 2, \cdots, s-1) \\ &\geqq \text{Minimum } [D(\Phi) - c\, D(\Phi)^{1/2}] \text{ for } \xi_i = \theta_i \qquad (i = 1, 2, \cdots, s-1) \\ &\geqq D(\theta_s) - c\, D(\theta_s)^{1/2} = \tau_s - c\, \tau_s^{1/2} \end{aligned}$$

$$l_s \gtrsim \frac{4\pi s}{\iint_{T'} \sqrt{\frac{\rho}{p}}\, d\sigma}.$$

We have the result

$$l_n \cong \frac{4\pi n}{\iint_{T'} \sqrt{\frac{\rho}{p}}\, d\sigma}.$$

Now we return to the system (E) and look for the extremal of

$$K(\Phi) = \iint_{T'} [\mathfrak{L}(\Phi)]^2\, d\sigma.$$

Let c represent a constant depending on a, b, f, p and ρ, then

$$\begin{aligned} \left| \iint_{T'} 2ap^{1/2}\Phi_X(\Delta\Phi)\, d\sigma \right| &\leqq c \left\{ \iint_{T'} \Phi_X^2\, d\sigma \iint_{T'} (\Delta\Phi)^2\, d\sigma \right\}^{1/2} \\ &\leqq c\, D(\Phi)^{1/2}\, H(\Phi)^{1/2} \\ \left| \iint_{T'} 2fp^{1/2}\Phi(\Delta\Phi)\, d\sigma \right| &\leqq c\, H(\Phi)^{1/2} \\ \left| \iint_{T'} 2ab\Phi_X\Phi_Y d\sigma \right| &\leqq c \iint_{T} (\Phi_X^2 + \Phi_Y^2)\, d\sigma \leqq c\, D(\Phi). \end{aligned}$$

Making similar estimates for the other terms in K we obtain

$$K(\Phi) = H(\Phi) + O(D^{1/2} H^{1/2}) + O(H^{1/2}) + O(D).$$

But

$$\sqrt{H} \geqq D - c\, D^{1/2} = (D^{1/2} - c/2)^2 - c^2/4$$
$$D^{1/2} = O(H^{1/4})$$

and

$$K(\Phi) = H(\Phi) + O(H^{3/4}).$$

The terms in the expression $O(H^{3/4})$ are algebraic, consequently the derivative dK/dH will be positive for H sufficiently large. This means that N exists such that for $H > N$, the minimum of $K(\Phi)$ is taken on when $H(\Phi)$ takes on its minimum.

Using $k(\xi_1, \xi_2, \cdots, \xi_{n-1})$ to denote the minimums of $K(\Phi)$ and remembering that

$$\lambda_n^2 = k(U_1, U_2, \cdots, U_{n-1})$$
$$l_n^2 = h(W_1, W_2, \cdots, W_{n-1}),$$

we have for n sufficiently large

$$l_n^2 - O(l_n^{3/2}) \leqq k(W_1, W_2, \cdots, W_{n-1}) \leqq k(U_1, U_2, \cdots, U_{n-1}) = \lambda_n^2$$
$$\lambda_n^2 \leqq h(U_1, U_2, \cdots, U_{n-1}) + O(h(U_1, U_2, \cdots, U_{n-1})^{3/4})$$
$$\leqq l_n^2 + O(l_n^{3/2}),$$

wherefore

$$\lambda_n \cong l_n \cong \frac{4\pi n}{\iint_{T'} \sqrt{\frac{\rho}{p}}\, dXdY}.$$

But $\rho = 1/\delta$, $p = \alpha^2\delta$, $\alpha = \sqrt{AC - B^2}$, $dxdy = (1/\delta)dXdY$, hence transforming to the variables x, y we have

$$\lambda_n \cong \frac{4\pi n}{\iint_T \frac{dxdy}{\sqrt{AC - B^2}}}.$$

Consideration of the system (B).

(B)
$$Lz + \sigma z = 0$$
$$[z]_R = 0.$$

The solutions $z_n(x, y)$ satisfy the integral equation

$$z_n(x, y) = \sigma_n \iint_T \Gamma(x, y, \xi, \eta) z_n(\xi, \eta)\, d\xi d\eta.$$

Hence the $\{\sigma_n\}$ are the zeros of the Fredholm determinant relative to the kernel $\Gamma(x, y, \xi, \eta)$. The C. V. λ_n are the zeros of Fredholm determinant relative to the kernels $\Gamma_1(x, y, \xi, \eta)$, $\Gamma_2(x, y, \xi, \eta)$ associated with $\Gamma(x, y, \xi, \eta)$.

Gheorghiu [4, p. 29] has proved that if the series $\sum_{n=1}^{\infty} (1/\lambda_n{}^{\alpha})$ converges, the series $\sum_{n=1}^{\infty} (1/|\sigma_n|^{\alpha})$ must converge, where of course the terms containing $\sigma_i = 0$ are omitted from the series. Setting $\alpha = 1 + \epsilon$ we have the result,

$$\sum_{n=1}^{\infty} (1/|\sigma_n|^{1+\epsilon})$$

converges for every $\epsilon > 0$.

PENNSYLVANIA STATE COLLEGE.

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GEGENBAUER FUNCTIONS WITH A NEGATIVE INDEX.

By MAURICE DE DUFFAHEL.

Gegenbauer polynomials, which play a rather important part in several problems (for instance in the potential theory and harmonic analysis), are defined by the expansion

$$(1 - 2\alpha x + \alpha^2)^{-\nu} = \sum_{n=0}^{\infty} \alpha^n C_n^{\nu}(x)$$

where ν is a positive number. $C_n^{\nu}(x)$ is a Gegenbauer polynomial of order n, with the superior index ν. It can be expressed (omitting a constant factor) by means of a Gauss hypergeometric series as

$$F(-n/2,\ n/2 + \nu;\ 1/2;\ x^2)$$

if n is even, with a similar formula when n is odd.

Let us now suppose that ν is a *negative* number. Writing $\nu = -\mu$, we see that the hypergeometric function,

$$F(-n/2,\ n/2 - \mu;\ 1/2;\ x^2),$$

is definite, even if μ is an integer; but, in this case, the expansion

$$(1 - 2\alpha x + \alpha^2)^{\mu}$$

becomes improper, giving rise to a limited number of terms. The polynomial corresponding to the hypergeometric function, and which we shall name $C_n^{-\nu}(x)$, cannot therefore be attached to the said expansion. Now the question is, which generating function shall we have for this new polynomial?

The answer is, $C_n^{-\nu}(x)$ or $C_n^{\mu}(x)$ comes from the generating function

$$(1 - 2\alpha x + \alpha^2)^{\mu} \log (1 - 2\alpha x + \alpha^2).$$

In order to establish this result, let us begin with $\mu = 1$. Starting from

$$(1 - 2\alpha x + \alpha^2) \log (1 - 2\alpha x + \alpha^2) = \Sigma\, \alpha^n P_n(x),$$

we find readily that P_n satisfies the recurrence-formulae

$$nP_n = xP'_n - P'_{n-1}$$
$$P'_{n+1} - xP'_n = (n - 2)\, P_n;$$

hence the differential equation

$$(1-x^2)P''_n + xP'_n + n(n-2)P_n = 0.$$

But the differential equation for Gegenbauer polynomial C_n^{ν} is

$$(1-x^2)y'' - (2\nu+1)xy' + n(n+\nu-1)y = 0$$

and we obtain the differential equation for P_n by taking in the latter $\nu = -1$. So, omitting a constant factor, we infer immediately that

$$P_n = C_n^{-1}(x).$$

Coming now to the general case, if we have

$$(1-2\alpha x+\alpha^2)^{\mu}\log(1-2\alpha x+\alpha^2) = \sum_n \alpha^n P_n^{\mu}(x),$$

we can write

$$(1-2\alpha x+\alpha^2)^{\mu+1}\log(1-2\alpha x+\alpha^2) = \Sigma\,(1-2\alpha x+\alpha^2)\alpha^n P_n^{\mu}(x);$$

hence

$$P_n^{\mu+1} = P_n^{\mu} - 2xP^{\mu}_{n-1} + P^{\mu}_{n-2}.$$

This being exactly one of Gegenbauer's recurrence formulae, we see that

$$P_n^2 = C_n^{-2}(x),$$

and so on.

* * * * *

This result can be used to find solutions of the partial differential equation with two variables

$$\Delta\cdots\Delta\Delta U = 0$$

or

$$\Delta^p U = 0,$$

where

$$\Delta U = \partial^2 U/\partial x^2 + \partial^2 U/\partial y^2.$$

For, if we write

$$u = x+iy \qquad v = x-iy,$$

the equation becomes

$$\partial^{2p} U/\partial u^p \partial v^p = 0$$

of which the general solution is

$$U(x,y) = V(u,v) = u^{p-1}F(v) + v^{p-1}G(u),$$

F and G being arbitrary functions.

Let us take

$$F(v) = v^{p-1} \log v, \qquad G(u) = u^{p-1} \log u,$$

so

$$U = u^{p-1} v^{p-1} \log uv,$$

or

$$U(x, y) = (x^2 + y^2)^{p-1} \log (x^2 + y^2).$$

This is a solution of $\Delta^p U = 0$, and it is clear that

$$\partial^m U / \partial x^m$$

is also a solution.

So also is the function

$$U_m(x, y) = \frac{(-1)^m}{m!} \frac{\partial^m}{\partial \alpha^m} \{[(x-\alpha)^2 + y^2]^{p-1} \log [(x-\alpha)^2 + y^2]\}.$$

But we can write, using Taylor's formula,

$$[(x-\alpha)^2 + y^2]^{p-1} \log [(x-\alpha)^2 + y^2] = \sum_m \alpha^m U_m(x, y).$$

Now suppose that we are on the circle

$$x^2 + y^2 = 1.$$

This formula becomes

$$(1 - 2\alpha x + \alpha^2)^{p-1} \log (1 - 2\alpha x + \alpha^2) = \Sigma \alpha^m U_m$$

so that U_m becomes identical to a Gegenbauer function with a negative index, here

$$C_m^{1-p}(x).$$

Gegenbauer functions with a negative index are therefore p-harmonic functions, in the xy-plane, and on the circle $x^2 + y^2 = 1$.

This fact is analogous to the fact of Legendre polynomials being harmonic functions, for space, on the sphere $x^2 + y^2 + \zeta^2 = 1$, and Hermite two-variable polynomials being harmonic functions, for hyperspace, on the hypersphere $x^2 + y^2 + \zeta^2 + z^2 = 1$.

STAMBOUL, TURQUIE D'EUROPE.

NOTE ON POTENTIAL THEORY IN n-SPACE.

By J. J. L. HINRICHSEN.

In this note the elegant elementary methods developed by Schmidt[1] and applied by him to Newtonian potentials are extended to obtain the properties near the acting masses of the integrals defining the potentials of simple surface, double surface, and volume distributions in n-space.

The integrals by means of which we shall define surface and volume potentials in n-space $(n > 2)$ are the following: for a simple distribution of density $f(x_1, \cdots, x_{n-1})$ on a surface α:

$$V(\xi_1, \cdots \xi_n) = \int_\alpha f(1/r^{n-2})\, d\sigma; \tag{1}$$

for a distribution of density $\rho(x_1, \cdots x_n)$ throughout a volume G:

$$A(\xi_1, \cdots \xi_n) = \int_G \rho(1/r^{n-2})\, d\tau; \tag{2}$$

and for a double surface distribution of moment $f(x_1, \cdots x_{n-1})$ on α:

$$W(\xi_1, \cdots \xi_n) = \int_\alpha f \frac{\partial}{\partial \nu}(1/r^{n-2})\, d\sigma. \tag{3}$$

Here $d\sigma$ and $d\tau$ represent the differential elements of surface and volume respectively, while the simple integral signs are used to denote the $(n-1)$- and n-fold multiple integrals. Also

$$r^2 = \sum_{i=1}^{n} (\xi_i - x_i)^2,$$

and $\partial/\partial\nu$ represents the derivative in the direction of the normal to α. Exterior to the acting masses, these integrals define analytic functions of $\xi_1, \cdots \xi_n$ which satisfy Laplace's equation in n-dimensions:

$$\nabla^2 u \equiv \sum_{i=1}^{n} (\partial^2 u / \partial \xi_i^2) = 0.$$

[1] E. Schmidt, "Bemerkung zur Potentialtheorie," pp. 364-383 of *Mathematische Abhandlungen H. A. Schwarz gewidmet*, Berlin, 1914. Methods related to those of Schmidt are used by M. M. Sullivan in a paper entitled "On the derivatives of Newtonian and logarithmic potentials near the acting masses," *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 137-171.

We shall investigate the behavior of these integrals as the point $(\xi_1, \cdots \xi_n)$ approaches the distribution.

Let us assume that the surface containing the distribution or bounding the volume containing the distribution is finite and integrable. We shall suppose that the surface point P in whose neighborhood the potential is to be investigated is a simple point and that the surface has a continuous normal at P as well as in a certain neighborhood of P. That is to say, we assume the possibility of cutting from the surface a piece α which by proper choice of rectangular coördinates may be represented by:

$$x_n = F(x_1, \cdots x_{n-1}), \tag{4}$$

where F, $\partial F/\partial x_i$ are continuous for $|x_i| \leqq a$, $a > 0$ and $i = 1, 2, \cdots (n-1)$. Also α shall contain P but not on its boundary, that is for P, $|x_i| < a$. Finally the remaining part of the surface after cutting out α cannot contain P either as a boundary or interior point. Since the potential at P of the remaining part of the surface is analytic, we need only investigate the potential of the distribution on α for the neighborhood of P.

We shall assume Green's theorem in n-dimensions:

$$\int_G \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}\, d\tau = -\int_\gamma \sum_{i=1}^{n} X_i \cos(\nu, x_i)\, d\sigma, \quad \text{(inner normal)}.$$

Then in a manner similar to that used for the Newtonian case, we can show if $\phi(x_1, \cdots x_n)$ is of class C''[2] throughout the region G and its boundary γ:

$$\int_G \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right) d\tau = -\int_\gamma \phi \frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right) d\sigma + \begin{cases} (n-2)\, \mathfrak{M}\, \phi(\xi_1, \cdots \xi_n) \\ \binom{n-2}{2} \mathfrak{M}\, \phi(\xi_1, \cdots \xi_n), \\ 0 \end{cases} \tag{5}$$

and if ϕ is of class C'',

$$\int_G \frac{1}{r^{n-2}} \nabla^2 \phi\, d\tau = -\int_\gamma \frac{1}{r^{n-2}} \frac{\partial \phi}{\partial \nu}\, d\sigma + \int_\gamma \phi \frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right) d\sigma - \begin{cases} (n-2)\, \mathfrak{M}\, \phi(\xi_1, \cdots \xi_n) \\ \binom{n-2}{2} \mathfrak{M}\, \phi(\xi_1, \cdots \xi_n). \\ 0 \end{cases} \tag{6}$$

The first, second or third value in the braces is to be chosen according to

[2] By a function of class $C^{(i)}$ we mean a function which is continuous together with its partial derivatives up to the i-th order inclusive.

whether P is a point interior to G, a boundary point, or an exterior point respectively. Here

$$\mathcal{H} = \begin{cases} \dfrac{2(2\pi)^{(n-1)/2}}{1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)} & \text{if } n \text{ is odd,} \\ \dfrac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot(n-2)} & \text{if } n \text{ is even.} \end{cases} \tag{7}$$

The lemmas corresponding to those given by Schmidt may be established for n-space by the same methods as for 3-space. It will be found that regardless of the position of the point P,

$$\int_T (1/r^{n-1})\,d\tau \leqq K_1 v^{1/n}$$

$$\int_T (1/r^{n-2})\,d\tau \leqq K_2 v^{2/n}$$

$$\int_S (1, r^{n-2})\,d\sigma \leqq K_3 J^{1/(n-1)},$$

where K_1, K_2, K_3 are constants dependent on n, v is the n-dimensional volume of the region T, and J denotes the $(n-1)$-dimensional volume of the projection of S on the $\xi_1, \xi_2, \cdots \xi_{n-1}$-space. By means of these lemmas, it is not difficult to show that *with a bounded density, the potential of a simple surface distribution* (1) *is everywhere a continuous function of the position of the point P even when the point lies on the surface or on its boundary.* Likewise the same procedure may be followed to show that *with a bounded density, the potential of a finite volume distribution* (2) *and its first derivative is everywhere continuous,—also at a point of discontinuity of the density and on the boundary of the volume,—and the first derivative is obtainable from the integral defining the potential by differentiating under the integral sign.*

Consider now the piece of surface α defined by (4) with a double spread of moment $f(x_1, \cdots x_{n-1})$. Let us suppose that the functions f and F are of class C'. Choose h so large that the entire piece of surface α lies between the two planes $x_n = \pm h$ without having points in common therewith. The space bounded by $x_n = \pm h$, $x_i = \pm a$ $(i = 1, 2, \cdots n-1)$, is then divided into two parts T_1 and T_2 by α. Let the positive normal on α point towards T_1 and γ_1 denote that part of the boundary of T_1 which is made up of planes bounding the n-dimensional parallelopiped.

We shall apply Green's theorem (5) to the region T_1 and the function $f(x_1, \cdots x_{n-1})$ which is defined inclusive of the boundaries of T_1. Upon introducing W defined by (3), we obtain

$$(8)\qquad W = -\int_{\gamma_1} f\,\frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right) d\sigma + \int_{T_1} \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}\,\frac{\partial}{\partial \xi_i}\left(\frac{1}{r^{n-2}}\right) d\tau + \begin{cases} (n-2)\,\mathcal{H} f(\xi_1, \cdots \xi_{n-1}) \\ \dfrac{n-2}{2}\,\mathcal{H} f(\xi_1, \cdots \xi_{n-1}). \\ 0 \end{cases}$$

On applying our earlier results to the integrals appearing in (8), we find that *the potential of a double distribution* (3) *has continuous limit values on both sides of the surface which from the side of the negative to the side of the positive normal has a break of* $(n-2)\,\mathcal{H} f$. *On the surface itself, the defining integral has the value of the arithmetic mean of these two limit values.*

Let us now suppose the function F to be of class C'' and f to be of class C'. Let Γ be the $p = (n-2)$-dimensional manifold bounding α. Then Stokes' theorem in n-space [3] may be written:

$$\int_\Gamma X_{s_1 \ldots s_p}\, d(x_{s_1} \cdots x_{s_p}) = \int_\alpha X_{s_1 \ldots s_{p+1}}\, d(x_{s_1} \cdots x_{s_{p+1}}),$$

where

$$X_{s_1 \ldots s_{p+1}} \equiv \frac{\partial X_{s_1 \ldots s_p}}{\partial x_{s_{p+1}}} - \frac{\partial x_{s_{p+1}, s_2 \ldots s_p}}{\partial x_{s_1}} - \frac{\partial X_{s_1 s_{p+1} s_3 \ldots s_p}}{\partial x_{s_2}} - \cdots - \frac{\partial X_{s_1 \ldots s_{p-1} s_{p+1}}}{\partial x_{s_p}}.$$

In the left member $s_1, \cdots s_p$ are umbral and $s_1 < s_2 < \cdots < s_p$, while in the right member $s_1, \cdots s_{p+1}$ are umbral and $s_1 < s_2 < \cdots < s_{p+1}$. Here $s_1, s_2, \cdots s_p$ are any p numbers chosen from $1, 2, \cdots n$ and

$$X_{s_1 \ldots s_i\, s_j \ldots s_p} = -X_{s_1 \ldots s_j\, s_i \ldots s_p}.$$

For our purpose it is convenient to define

$$X_{s_1 s_2 \ldots s_p} = \begin{cases} 0 & \text{if } s_1, \cdots s_p \text{ contains } j, \\ (-1)^{\bar{s}_p + j} f \cos(\nu, x_{\bar{s}_p})\,(1/r^p) & \text{if } s_1, \cdots s_p \text{ does not contain } j \text{ and } \bar{s}_p > j, \\ (-1)^{\bar{s}_p + j - 1} f \cos(\nu, x_{\bar{s}_p})\,(1/r^p) & \text{if } s_1, \cdots s_p \text{ does not contain } j \text{ and } \bar{s}_p < j. \end{cases}$$

Here $j, s_1, \cdots s_p, \bar{s}_p$ contains $1, 2, \cdots n$, and (ν, x_{s_p}) represents the angle between the normal to α and the x_{s_p}-axis. Upon substituting, we find it necessary to consider two cases, 1) n odd and 2) n even. Here as well as later we shall denote a function which is analytic provided that the point P does not lie on the boundary Γ of the piece of surface α by R.[4]

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[3] We shall adopt the notation of F. D. Murnaghan, *Vector Analysis and the Theory of Relativity* (1922), p. 33. Also see H. Poincaré, "Sur les résidues des intégrales doubles," *Acta Mathematica*, vol. 9 (1887), p. 321.

[4] Even though two functions are not identical, if they have the above-mentioned property, they will both be denoted by R.

For case 1), we find

$$\frac{\partial V}{\partial \xi_j} = \int_a f_j \frac{1}{r^{n-2}}\, d\sigma - \int_a f \cos(\nu, x_j)\, \frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right) d\sigma - R, \tag{9}$$

where

$$f_j = \sum_{i=1}^{n} \{\cos(\nu, x_i)\, \frac{\partial}{\partial x_j}\, [f \cos(\nu, x_i)] - \cos(\nu, x_j)\, \frac{\partial}{\partial x_i}\, [f \cos(\nu, x_i)]\}.$$

For the second case it is convenient to re-define the function f by setting

$$\bar{f} = f \sum_{i=1}^{n} (-1)^i \cos^2(\nu, x_i),$$

and then to remove the bar from the new function. We so obtain:

$$\frac{\partial V}{\partial \xi_j} = \int_a \bar{f}_j \frac{1}{r^{n-2}}\, d\sigma - \int_a \tilde{f}_j \cos(\nu, x_j)\, \frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right) d\sigma - R, \tag{9'}$$

where

$$\bar{f}_j = \sum_{i=1}^{n} \left\{ (-1)^i \cos(\nu, x_i) \frac{\partial}{\partial x_j} \left(\frac{f \cos(\nu, x_i)}{\sum_{i=1}^{n} (-1)^i \cos^2(\nu, x_i)} \right) + (-1)^{j+1} \cos(\nu, x_j) \frac{\partial}{\partial x_i} \left(\frac{f \cos(\nu, x_i)}{\sum_{i=1}^{n} (-1)^i \cos^2(\nu, x_i)} \right) \right\}$$

and

$$\tilde{f}_j = \frac{(-1)^j f}{\sum_{i=1}^{n} (-1)^i \cos^2(\nu, x_i)}.$$

Applying our previous results to (9) and (9'), we find that *the derivatives* $\partial V/\partial \xi_j$ $(j = 1, 2, \cdots n)$, *have continuous limiting values on both sides of the surface* α. *The formulas for the break in limit values take on a simple form when the* x_n*-axis is chosen parallel to the normal through the point. Then we find* $\partial V/\partial \xi_j$ $(j = 1, 2, \cdots n-1)$, *to be continuous while* $\partial V/\partial \xi_n$ *has a break of* $-(n-2)\, \mathfrak{M} f$ *in going from the side of the negative to the side of the positive normal.*

Now let the functions F and f be of class C''. If we apply Green's theorem (6) to the region T_1 and the function $f(x_1, \cdots x_{n-1})$ defined in T_1 inclusive of its boundaries, we obtain

$$W = \int_{T_1} \frac{1}{r^{n-2}} \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial x_i^2}\, d\tau + \int_a \frac{1}{r^{n-2}} \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} \cos(\nu, x_i)\, d\sigma + R + \begin{cases} (n-2)\, \mathfrak{M} f(\xi_1, \cdots \xi_{n-1}) \\ \frac{n-2}{2}\, \mathfrak{M} f(\xi_1, \cdots \xi_{n-1}). \\ 0 \end{cases} \tag{10}$$

Upon analyzing the integrals in (10), we conclude that *the derivatives* $\partial W/\partial \xi_j$ $(j = 1, 2, \cdots n)$, *have continuous limit values on both sides of* α. *The formulas for the break in limit values take on a simple form when the* x_n*-axis is chosen parallel to the normal through the point. Then we obtain:* $(n-2)\,\mathcal{H}\,(\partial f/\partial x_j)$ $(j = 1, 2, \cdots n-1)$, *in going from the side of the negative to the side of the positive normal. The derivative in the direction of the normal is continuous.*

Let G be any region in n-space satisfying the conditions of Green's theorem. Let the boundary γ consist of γ_1 and α defined by (4) and let the density $\rho(x_1, \cdots x_n)$ be of class C' interior to and on the boundary of G. Then if A is defined by (2), it may be shown that

$$\frac{\partial A}{\partial \xi_j} = \int_G \frac{\partial \rho}{\partial \xi_j} \frac{1}{r^{n-2}}\, d\tau + \int_\alpha \rho \cos(\nu, x_j) \frac{1}{r^{n-2}}\, d\sigma + R. \tag{11}$$

We are now prepared to state the three general theorems.

THEOREM 1. *If* F *is of class* $C^{(k)}$, ρ *is of class* $C^{(k-1)}$, *then* A *together with its partial derivatives up to the* k*-th order inclusive are continuous both inside and outside of* G *and have continuous limit values on both sides of the bounding surface* α.

THEOREM 2. *If* F *is of class* $C^{(k+1)}$, f *is of class* $C^{(k)}$, *then* V *together with its partial derivatives up to the* k*-th order inclusive have continuous limit values on both sides of the surface* α.

THEOREM 3. *If* F *is of class* $C^{(k+1)}$, f *is of class* $C^{(k+1)}$, *then* W *together with its partial derivatives up to the* k*-th order inclusive have continuous limit values on both sides of the surface* α.

We have already outlined the proofs of these three theorems for $k = 1$. By means of mathematical induction and the identities (11), (9), (9′) and (10) we can show that these theorems are true for all values of k.

The formulas (11), (9) and (10) also yield the results for the successive breaks. For example, from (11) we may read off the breaks in the $(n+1)$-th-order derivatives of the potential of a volume distribution on the boundary surface in terms of the breaks in the n-th-order derivatives of the potential of a volume distribution and the breaks in the n-th-order derivatives of the potential of a simple surface distribution. Equations (9) and (9′) give the breaks on the surface in the $(n+1)$-th-order derivatives of the potential of a simple surface distribution in terms of the breaks in the n-th-order derivatives

of the potential of a simple surface distribution and the breaks in the n-th-order derivatives of the potential of a double distribution. Finally from the breaks in the $(n+1)$-th-order derivatives of the potentials due to a volume distribution and the breaks in the $(n+1)$-th-order derivatives of the potential due to a simple surface distribution, we can find the break on the surface of the $(n+1)$-th-order derivatives of the potential due to a double surface distribution.

Thus for example, if the x_n-axis is chosen parallel to the normal to the surface at P, we obtain the following breaks in the second-order derivatives $\partial^2 A/\partial\xi_j\xi_k$:

$$\begin{array}{ll} 0 & \text{if } k \neq n, \\ -(n-2)\,\mathcal{H}\,\rho\cos(\nu, x_j) & \text{if } k = n. \end{array}$$

If we define $\alpha_{jk} = (\partial/\partial x_k)(\cos(\nu, x_j))$, then $\alpha_{jn} = 0$ and the breaks in $\partial^2 V/\partial\xi_j\partial\xi_k$ are found to be:

$$\begin{array}{ll} -(n-2)\,\mathcal{H} f\alpha_{jk} & \text{if } j, k \neq n \text{ and } n \text{ is odd}, \\ (-1)^{(j+1)}(n-2)\,\mathcal{H} f\alpha_{jk} & \text{if } j, k \neq n \text{ and } n \text{ is even}, \\ -(n-2)\,\mathcal{H}\,(\partial f/\partial x_j) & \text{if } j \neq k = n, \\ (n-2)\,\mathcal{H} f \sum_{i=1}^{n-1} \alpha_{ii} & \text{if } j = k = n. \end{array}$$

Likewise the breaks in $\partial^2 W/\partial\xi_j\partial\xi_k$ may be obtained:

$$\begin{array}{ll} (n-2)\,\mathcal{H}\,(\partial^2 f/\partial x_j\partial x_k) & \text{if } j, k \neq n, \\ -(n-2)\,\mathcal{H} \sum_{i=1}^{n-1} (\partial f/\partial x_i)\alpha_{ij} & \text{if } j \neq k = n, \\ -(n-2)\,\mathcal{H} \sum_{i=1}^{n-1} (\partial^2 f/\partial x_i^2) & \text{if } j = k = n. \end{array}$$

Finally from the breaks in $\partial^2 A/\partial\xi_j^2$, we find that for an interior point of the region G,

$$\nabla^2 A = -(n-2)\,\mathcal{H}\,\rho$$

which is Poisson's equation in n-dimensions. It is to be noted that for $n = 3$ these results reduce to those given by Schmidt.

IOWA STATE COLLEGE.

THE GENERALIZED DOUBLE PENDULUM.

By G. Baley Price.

Introduction. The dynamical system formed by suspending a simple pendulum from the bob of another simple pendulum and allowing the two to move in a vertical plane under the attraction of gravity is called a double pendulum [3 [1]; no results of this paper are used in the present treatment of the problem]. The following obvious generalization will be called a generalized double pendulum: a heavy particle of mass m_1 moves on a fixed, simple closed curve C_1 in a vertical plane and carries without rotation and in the same vertical plane a second simple closed curve C_2, on which a second heavy particle of mass m_2 moves. The meanings of the terms n-pendulum and generalized n-pendulum are clear from these definitions also.

The purpose of this note is (1) to use the generalized n-pendulum to illustrate certain results in the general theory of reversible dynamical systems, and (2) to establish in the large a result for the generalized double pendulum which so far in the general theory has been established only in the small (see section 3 below).

1. *The equations of motion.* Take axes in the plane of C_1 and C_2 with the ξ_2-axis directed downward and the ξ_1-axis to the right. The curves C_1, C_2 are defined parametrically by

$$(1) \qquad C_1 : \xi_i = x_i(u), \qquad C_2 : \xi_i = y_i(v), \qquad (i = 1, 2),$$

where $x_i(u)$, $y_i(v)$ are analytic and periodic with periods ω_u, ω_v. Furthermore, u and v are arc lengths on C_1 and C_2:

$$(2) \qquad (x_u x_u) \equiv 1, \qquad (y_v y_v) \equiv 1,$$

where the notation is explained by

$$(3) \qquad (w_u z_v) \equiv \frac{\partial w_1}{\partial u}\frac{\partial z_1}{\partial v} + \frac{\partial w_2}{\partial u}\frac{\partial z_2}{\partial v}.$$

The coördinates of m_1 and m_2 are respectively

$$\xi_i = x_i(u), \qquad \xi_i = x_i(u) + y_i(v), \qquad (i = 1, 2).$$

[1] Numbers in brackets refer to articles cited in the bibliography at the end of this note.

Then T and U are

$$(4)\qquad \begin{aligned} T &= [(m_1+m_2)/2][u'^2 + 2m(x_u y_v)u'v' + mv'^2], \\ U &= (m_1+m_2)g[x_2(u) + my_2(v)], \end{aligned}$$

where $m = m_2/(m_1+m_2)$. Then the equations of motion in the Lagrangian form give

$$(5)\qquad \begin{aligned} u'' &= \frac{m(x_u y_v)(x_{uu}y_v)u'^2 - m(x_u y_{vv})v'^2 + g[x_u^{(2)} - m(x_u y_v)y_v^{(2)}]}{1 - m(x_u y_v)^2}, \\ v'' &= \frac{-(x_{uu}y_v)u'^2 + m(x_u y_{vv})(x_u y_v)v'^2 + g[y_v^{(2)} - (x_u y_v)x_u^{(2)}]}{1 - m(x_u y_v)^2}. \end{aligned}$$

Now $(x_u y_v)$ is the scalar product of two unit vectors by (2) and (3); then since $0 < m < 1$, we see that the denominators in (5) are positive. Then the equations of motion for the generalized double pendulum are analytic for all values of (u, v).

2. *Application of general theory.* The generalized double pendulum is a reversible dynamical system with two degrees of freedom; we shall show now how certain results in the general theory of such systems can be applied to it.

There are two characteristic surfaces associated with the problem. The first of these is given by [5, equation (7)]. The system can be interpreted as the motion on this surface of a particle of unit mass under the action of forces derived from U in (4). The equations of this surface in $(x_1, \cdots, x_4)$ space are

$$x_i = m_1^{1/2}x_i(u), \qquad x_{i+2} = m_2^{1/2}[x_i(u) + y_i(v)],$$

where $(i = 1, 2)$. Since $x_i(u)$, $y_i(v)$ are periodic, the first characteristic surface is homeomorphic to a torus. For the ordinary double pendulum it can be shown that this surface is a surface of revolution.

The second characteristic surface enters through the Principle of Least Action. It is defined by

$$(6)\qquad ds^2 = (U+h)T\,dt^2,$$

and the orbits for a given value of h can be interpreted as geodesics on this surface [2, pp. 36-39]. Now for h sufficiently large, $(U+h)$ is positive for all values of (u, v); then since (6) is invariant under the group of transformations

$$\bar{u} = u + n_1\omega_u, \qquad \bar{v} = v + n_2\omega_v,$$

where n_1, n_2 are any integers, we may consider that the second characteristic

surface is closed and homeomorphic to a torus. Then if the energy of the generalized double pendulum is large enough so that motion is possible over the entire first characteristic surface, the minimum and minimax methods [1, Part II] and their extensions [4] establish immediately the existence of an infinite number of closed periodic orbits.

Now the generalized double pendulum has a certain number of positions of equilibrium corresponding to a point on C_1 with one on C_2 at which the tangents to these curves are horizontal. There are at least four positions of equilibrium since each curve has at least one maximum and one minimum.

Assume now that

$$(7) \qquad (x_u^{(2)})_0 = 0, \quad (x_{uu}^{(2)})_0 < 0, \quad (y_v^{(2)})_0 = 0, \quad (y_{vv}^{(2)})_0 < 0,$$

where the subscript zeros indicate that the derivatives are evaluated for $u = v = 0$. Then C_1 and C_2 have maxima for $u = v = 0$, and $(0, 0)$ is a position of stable equilibrium. We shall assume also that the parameters (u, v) are chosen in such a way that they increase as C_1, C_2 are described in the positive sense, where the rotation which carries the positive ξ_1-axis into the positive ξ_2-axis is positive. Then from (2) and (7) we have $(x_u y_v)_0 = 1$, and the expansions of T, U about $(0, 0; 0, 0)$ are

$$(8) \qquad \begin{aligned} T &= [(m_1 + m_2)/2][u'^2 + 2mu'v' + mv'^2] + \cdots, \\ U &= [(m_1 + m_2)g/2][(x^{(2)})_0 u^2 + m(y^{(2)})_0 v^2] + \cdots. \end{aligned}$$

Then it is possible to introduce principal coördinates [6, chap. VII] (q_1, q_2) so that T, U become

$$(9) \qquad \begin{aligned} T &= (q'_1{}^2 + q'_2{}^2)/2 + \cdots, \\ U &= (\lambda_1{}^2 q_1{}^2 + \lambda_2{}^2 q_2{}^2)/2 + \cdots, \end{aligned}$$

where

$$(10) \qquad \left.\begin{matrix} \lambda_1{}^2 \\ \lambda_2{}^2 \end{matrix}\right\} = g \, \frac{[(x_{uu}^{(2)})_0 + (y_{vv}^{(2)})_0] \pm [[(x_{uu}^{(2)})_0 + (y_{vv}^{(2)})_0]^2 - 4(1-m)(x_{uu}^{(2)})_0 (y_{vv}^{(2)})_0]^{1/2}}{2(1-m)}.$$

Since (7) holds and $0 < m < 1$, we see that neither λ_1 nor λ_2 is zero, that both are pure imaginary, and that in general

$$(11) \qquad \lambda_1 \neq \lambda_2.$$

It is now possible to apply the results of [5]. We introduce the parameter μ as in [5, section 3]. Since λ_1, λ_2 are pure imaginary, the limiting integrable system $\mu = 0$ has two fundamental periodic orbits $O_1{}^0$, $O_2{}^0$. It follows from

(11) and [5, section 5, Corollary 1] that at least one of these orbits can be continued analytically for $\mu > 0$. Also the surface of section can be continued analytically, and the results of [5, section 10] can be applied. As long as the surface of section SS exists and the region of motion R is homeomorphic to a circular disc, there are at least two periodic orbits which join two points of the oval of zero velocity Z. Poincaré's Last Geometric Theorem can be applied at least if both O_1^0 and O_2^0 can be continued analytically; the rotation numbers α_B and α_P are functions of μ and the curves C_1, C_2, and we may conclude that at least for $\mu > 0$ but small there exists an infinite number of closed periodic orbits.

Suppose now that the curves C_1, C_2 have the symmetry specified by the hypotheses

$$(12) \qquad \begin{aligned} x_1(-u) &= -x_1(u), & y_1(-v) &= -y_1(v), \\ x_2(-u) &= x_2(u), & y_2(-v) &= y_2(v). \end{aligned}$$

Then from (4) we find that the system is symmetric in the origin on the characteristic surface. Furthermore, this symmetry is retained when principal coördinates are introduced; hence, the results of [5, section 11] can be applied. The ordinary double pendulum has the symmetry (12).

There are also positions of equilibrium at which $x_{uu}^{(2)}$ and $y_{vv}^{(2)}$ have opposite signs. Then (10) shows that one of the multipliers λ_1^2, λ_2^2 is positive and the other negative; hence, the limiting system $\mu = 0$ has one periodic orbit, which can be continued analytically for $\mu > 0$. If $x_{uu}^{(2)}$ and $y_{vv}^{(2)}$ are both positive, then λ_1^2, λ_2^2 are positive, and the limiting system has no periodic orbits.

The theorems of [5, section 5] yield entirely similar results for the generalized n-pendulum.

3. *On the existence of a periodic orbit joining two points of the oval of zero velocity.* The following question naturally arises: can the periodic orbits O_1, O_2 [5, section 10] be continued indefinitely, or do they exist in the small only? We shall not attempt to answer this question, but we shall establish the existence in the large of at least one periodic orbit joining two points of the oval of zero velocity Z for the symmetric generalized double pendulum.

First we introduce the notation

$$(w_u z_v)^* = \begin{vmatrix} \dfrac{dw_1}{du} & \dfrac{dz_1}{dv} \\ \dfrac{dw_2}{du} & \dfrac{dz_2}{dv} \end{vmatrix}.$$

Then from the first relation in (2) we find that

$$[y_v^{(2)} - (x_u y_v) x_u^{(2)}] = x_u^{(1)} (x_u y_v)^*.$$

Then the second equation in (5) becomes

$$v'' = \frac{-(x_{uu}y_v)u'^2 + m(x_u y_{vv})(x_u y_v)v'^2 + g x_u^{(1)}(x_u y_v)^*}{1 - m(x_u y_v)^2}. \tag{13}$$

We now assume that our generalized double pendulum is symmetric in the origin, i. e., we assume that (12) holds. Then there is a position of equilibrium at $(0, 0)$, and we assume that it is stable as specified by (7). Let G be defined as the connected region of the (u, v) plane about the origin in which the following hypotheses are satisfied:

$$(x_{uu}x_{uu}) > 0, \qquad (y_{vv}y_{vv}) > 0; \tag{14}$$

$$x_u^{(1)} < 0, \qquad y_v^{(1)} < 0; \tag{15}$$

$$x_u^{(2)} = 0 \quad \text{only for} \quad u = 0, \qquad y_v^{(2)} = 0 \quad \text{only for} \quad v = 0; \tag{16}$$

$$(x_u y_v) > 0. \tag{17}$$

All of these conditions are satisfied at $(0, 0)$, and there exists a connected region G containing the origin in which they are satisfied.

The tangents to C_1, C_2 are t_1: (x_u), t_2: (y_v). Let θ be the angle from t_1 to t_2. Then $(x_u y_v) = \cos\theta$ and $(x_u y_v)^* = \sin\theta$. Furthermore, we find from (2) that the vectors n_1, n_2 with the direction cosines $(x_{uu})/(x_{uu}x_{uu})^{\frac{1}{2}}$, $(y_{vv})/(y_{vv}y_{vv})^{\frac{1}{2}}$ are normal to C_1, C_2.

LEMMA 1. *In the region G*

$$\frac{(x_{uu}y_v)}{(x_{uu}x_{uu})^{\frac{1}{2}}} = -\frac{(x_u y_{vv})}{(y_{vv}y_{vv})^{\frac{1}{2}}} = (x_u y_v)^* = \sin\theta.$$

The ξ_2-axis is directed downward, but we have agreed to consider the rotation positive which carries the positive ξ_1-axis into the positive ξ_2-axis. Then at $(0, 0)$ we find that the angles from t_1 to n_1 and t_2 to n_2 are $\pi/2$. Then because of (2) and (14) these angles are $\pi/2$ throughout G. Then the angle from n_1 to t_2 is $(\theta - \pi/2)$, and $(x_{uu}y_v)/(x_{uu}x_{uu})^{\frac{1}{2}} = \cos(\theta - \pi/2)$. Also the angle from t_1 to n_2 is $(\theta + \pi/2)$, and $(x_u y_{vv})/(y_{vv}y_{vv})^{\frac{1}{2}} = \cos(\theta + \pi/2)$. The lemma follows.

Lemma 1 enables us to write the equations of motion (5), (13) in G in the form

$$
\begin{aligned}
u'' &= \frac{m \sin\theta[(x_{uu}x_{uu})^{1/2} \cos\theta u'^2 + (y_{vv}y_{vv})^{1/2} v'^2] - g[m \cos\theta y_v^{(2)} - x_u^{(2)}]}{1 - m\cos^2\theta}, \\
v'' &= \frac{-[(x_{uu}x_{uu})^{1/2}u'^2 + m(y_{vv}y_{vv})^{1/2} \cos\theta v'^2 - g x_u^{(1)}] \sin\theta}{1 - m\cos^2\theta}.
\end{aligned} \tag{18}
$$

LEMMA 2. *In the region G*

$$
\frac{(x_{uu}y_v)^*}{(x_{uu}x_{uu})^{1/2}} = \frac{(y_{vv}x_u)^*}{(y_{vv}y_{vv})^{1/2}} = -\cos\theta.
$$

This lemma is proved by straightforward methods, using the facts stated in the proof of Lemma 1 about the angles between n_i and t_i.

LEMMA 3. *The part of the curve $\sin\theta \equiv (x_u y_v)^* = 0$ which lies in G consists of a single branch whose slope is everywhere positive, and which passes through the origin and divides G into two parts.*

The fact that the curve $\sin\theta = 0$ passes through the origin follows from (7). The slope of $\sin\theta = 0$ is $dv/du = -(x_{uu}y_v)^*/(x_u y_{vv})^*$, and by Lemma 2 this is

$$
\frac{dv}{du} = \frac{(x_{uu}x_{uu})^{1/2}}{(y_{vv}y_{vv})^{1/2}}; \tag{19}
$$

hence, in G the slope is always positive. Because of (15) and (16) the curve crosses the v-axis at a single point only in G, and this crossing is at the origin. The curve is analytic at every point of G and has no multiple points. The single branch through the origin therefore extends into the first and third quadrants and continues with a positive slope until it passes out of G.

THEOREM 1. *If h is chosen sufficiently small so that the region of motion R is homeomorphic to a circular disc and lies in G, every orbit crosses the curve $\sin\theta = 0$ an infinite number of times.*

First, we notice from (17) that $-\pi/2 < \theta < \pi/2$ in G. Furthermore, $\sin\theta$ is positive in G above the curve $\sin\theta = 0$ and negative below it. Then since (15) holds, and since R lies in G, we see from (18) that $v'' = N \sin\theta$, where N is a function which is negative at all points of R. The theorem follows.

Assume henceforth that h is chosen so that the hypotheses of Theorem 1 are satisfied. Then R lies in G, contains the origin O, and is bounded by a

simple closed curve Z. Let X be the point in the third quadrant at which $\sin\theta = 0$ crosses Z, and Y the point at which the positive v-axis crosses Z. Let the region of R defined by $u \leqq 0$, $\sin\theta \geqq 0$ be designated by S. Then S is bounded by the segment OY of the v-axis, the arc XY of Z, and the arc OX of $\sin\theta = 0$.

LEMMA 4. *In the region S*

$$-g[m\cos\theta y_v^{(2)} - x_u^{(2)}] \geqq 0,$$

and the equality holds only at the origin.

From (7) and (16) we see that $x_u^{(2)} > 0$ for $u < 0$ and $y_v^{(2)} < 0$ for $v > 0$. By (17), $\cos\theta > 0$ everywhere in G. Then the desired inequality holds at all points of S in the second quadrant including the positive v-axis and the negative u-axis except for the origin, at which the equality holds by (7). It remains therefore to establish the inequality in the part of S in the third quadrant.

Now since $y_v^{(2)} > 0$ for $v < 0$, and since $0 < m < 1$, we have

$$-[my_v^{(2)}\cos\theta - x_u^{(2)}] > -[y_v^{(2)}\cos\theta - x_u^{(2)}]$$

for $v < 0$. Now by using (2) we find that

$$[y_v^{(2)}\cos\theta - x_u^{(2)}] = y_v^{(1)}\sin\theta.$$

Then since $y_v^{(1)} < 0$ in G by (15), and since $\sin\theta > 0$ in S except on the boundary OX, where it vanishes, the desired inequailty follows.

From Lemma 4 and (18) we see that in S we can write $u'' = M$, where $M \geqq 0$, the equality holding only at the origin. In S, therefore, the equations of motion take the form

$$\begin{aligned} u'' &= M, \\ v'' &= N\sin\theta. \end{aligned} \tag{20}$$

THEOREM 2. *If h is so chosen that R is homeomorphic to a circular disc and lies in G, there exists at least one periodic orbit which joins two points of Z; the two points of Z lie in the second and fourth quadrants respectively, and the orbit crosses $\sin\theta = 0$ at the origin and only at this point.*

Consider the orbit which at time $t = 0$ passes through an interior point P of the arc XY of Z. As long as it remains in S, the equations of motion can be written in the form (20). Then

$$u' = \int_0^t M\,dt, \qquad v' = \int_0^t N \sin\theta\,dt,$$

and $u' > 0$, $v' < 0$ for $t > 0$ and thereafter until the orbit has passed out of S across XOY at Q. The slope of this orbit is therefore negative in S except possibly at P, and it is surely negative at Q. Now by Lemma 3 the slope of XO is everywhere positive; hence, the orbit is not tangent to XOY at Q, and Q varies analytically with P.

Let r be a coördinate which specifies the position of P on Z, r varying from 0 to 1 as P varies from X to Y. Also let s be a coördinate which specifies the position of Q on XOY, s varying from 0 to 1 as Q varies from X to Q, and from 1 to 2 as Q varies from O to Y. Then as shown above, $s = f(r)$, where $f(r)$ is analytic for $0 < r < 1$. Since the slope of the orbit is everywhere negative in S except possibly at P, we find $\lim s = 0, 2$ as r approaches 0, 1 respectively. Then since the curve $s = f(r)$, $0 < r < 1$, is continuous, it has at least one intersection with the line $s = 1$.

We have thus shown that there is at least one point P, say P_1, for which the corresponding Q is at the origin O. Now the system is symmetric in the origin as specified by (12) by hypothesis; hence, the image in the origin of the orbit which joins P_1 to O is also an orbit OP_2 [5, section 2, Theorem 4], and the continuation of P_1O. Then P_1OP_2 is a periodic orbit which joins two points of Z [5, section 2, Theorem 3] and crosses $\sin\theta = 0$ at the origin and only at this point. Since the slope of P_1O is everywhere negative except possibly at P_1, we see that P_1 is in the second quadrant and P_2 in the fourth. Also the orbit is without double points. The proof of the theorem is complete.

The theorem which we have just proved does not show that the periodic orbits belong to a group of periodic orbits obtained by analytic continuation with μ (which is essentially the energy constant) as the parameter. Certain results in the large on the analytic continuation of periodic orbits are known [7, 8, 9], but they do not suffice to prove the existence of a periodic orbit with unrestricted variation of μ.

THE UNIVERSITY OF ROCHESTER,
ROCHESTER, NEW YORK.

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SPACE CURVES BELONGING TO A NON-SPECIAL LINEAR LINE COMPLEX.

By C. R. Wylie, Jr.

Introduction. By a "curve belonging to a linear line complex" is meant a curve all of the tangents to which belong to a linear line complex. Such curves exist, and have been the subject of a number of papers.

In his Doctor's Dissertation in 1877, Picard [1] cited an example of a sextic of genus one which belonged to a linear complex. In 1892, however, Steinmetz [2] erroneously concluded that curves of a complex had to be rational and moreover could have no cusps. Then in 1907, Snyder [3] demonstrated a 1 : 1 correspondence between curves belonging to a linear complex and curves lying on a quadric cone. Since curves of every genus can be found on a quadric cone this result confirmed Picard and refuted Steinmetz. Finally in 1925, Loria [4] reproduced the argument and conclusions of Steinmetz without perceiving the error.

From this point the present paper proceeds to an alternative development of the mapping relation between curves of a complex and curves on a quadric cone, derives the general parametric equations of such curves, deduces certain of their metrical properties, cites examples of curves of every genus which belong to a complex, proves that such curves may have any preassigned self-dual singularity, studies the $n-3 : n-3$ correspondence on complex curves with particular reference to a certain pair of sextics, and treats of the two families of rational sextics having six inflexions, those which belong to a linear complex and those which do not.

In order to find the parametric equations of complex curves it is necessary to have equations whereby the point common to two consecutive intersecting line can be found when the Plücker coördinates of the lines are given as functions of a single parameter. These equations are

[1] E. Picard, "Application de la Théorie des Complexes Linéaires a l'Étude des Surfaces et des Courbes Gauches," *Annales de l'École Normale*, 1877.

[2] C. P. Steinmetz, "Curves which are self-reciprocal in a linear nul-system," *American Journal of Mathematics*, vol. 14 (1892), pp. 161-185.

[3] V. Snyder, "Twisted curves whose tangents belong to a linear complex," *American Journal of Mathematics*, vol. 29 (1907), pp. 279-288.

[4] G. Loria, *Curve Sghembe Speciali*, vol. 2 (1925), p. 21.

$$A: \quad \begin{aligned} \rho x_1 &= P_{12}P'_{13} - P_{13}P'_{12} \\ \rho x_2 &= P_{12}P'_{23} - P_{23}P'_{12} \\ \rho x_3 &= P_{13}P'_{23} - P_{23}P'_{13} \\ \rho x_4 &= P_{14}P'_{23} + P_{42}P'_{13} + P_{34}P'_{12} \end{aligned} \qquad \text{where } P_{ik} = P_{ik}(t),\ P'_{ik} = \frac{dP_{ik}(t)}{dt}\ .$$

1. It is well known from line geometry that there is a 1 : 1 correspondence between the lines of S_3 and the points of a hyperquadric V_4^2 of S_5. Moreover the surface of tangents to a curve in S_3 is represented in S_5 by a curve which lies with its tangents on V_4^2. If the curve in S_3 belongs to a linear complex which is not special its image in S_5 lies with its tangents in the section of V_4^2 by a hyperplane S_4 which is not tangent to V_4^2. If in such a hyperplane the image curve is projected from a point on the V_3^2, intersection of V_4^2 and S_4, onto an S_3 there is obtained a space curve whose tangents all meet a fixed conic, and whose osculating planes therefore are all tangent to the same conic. This conic is the image in the projection of the quadric cone cut from V_3^2 by the tangent S_3 at the center of projection. By duality in space this configuration is sent into a curve lying on a quadric cone.

To find the equations of curves of a complex let

$$V_4^2 \equiv P_{12}P_{34} + P_{13}P_{42} + P_{14}P_{23} = 0$$

be the Plücker identity and

$$S_4 \equiv P_{13} + 2P_{42} = 0$$

be the linear complex containing the original curve. Then

$$V_3^2 \equiv P_{12}P_{34} - \tfrac{1}{2}P^2_{13} + P_{14}P_{23} = 0$$

and if the center of projection is

$$P_{23} = 1, \quad P_{12} = P_{13} = P_{14} = P_{34} = 0$$

and

$$P_{23} = 0$$

the S_3 of the projection, the conic, image of the cone of singular elements is

$$\begin{cases} 2P_{12}P_{34} - P^2_{13} = 0 \\ P_{14} = 0 \end{cases} \quad \text{or} \quad P_{12} : P_{13} : P_{34} : P_{14} = \tfrac{1}{2}t^2 : t : 1 : 0.$$

Then

$$\tfrac{1}{2}t^2P_{34} - t\,P_{13} + P_{12} - f(t)P_{14} = 0$$

represents a family of planes all of which are tangent to this conic. The parametric point equations of the cuspital edge of this family of planes are the Plücker coördinates of the tangents to the original curve. These are

$$
\begin{aligned}
P_{12} &= f - t f' + \tfrac{1}{2} t^2 f'' \\
P_{13} &= -f' + t f'' \\
P_{34} &= f'' \\
P_{14} &= 1 \\
P_{42} &= \frac{-f' + t f''}{-2} \\
P_{23} &= -ff'' + \tfrac{1}{2}\overline{f''}^2
\end{aligned}
$$

whence from equations A the parametric point equations of the original curve are

$$
B: \quad \begin{aligned}
x_1 &= -t \\
x_2 &= f - \tfrac{1}{2} t f' \\
x_3 &= -f' \\
x_4 &= 1.
\end{aligned}
$$

If f is an implicit function of t so that $\phi(f, t) = 0$ these equations become

$$
C^5: \quad \begin{aligned}
x_1 &= -t\, \partial\phi/\partial f \\
x_2 &= f\, \partial\phi/\partial f + \tfrac{1}{2} t\, \partial\phi/\partial t \\
x_3 &= \partial\phi/\partial t \\
x_4 &= \partial\phi/\partial f.
\end{aligned}
$$

In each case the genus of the curve is the genus of the function connecting f and t.

2. The parametric plane equations of these curves each contain the factor f''', hence $f''' = 0$ is the condition for an inflexion. The condition for stationary planes reduces to $\overline{f'''}^2 = 0$. Hence, *The stationary planes of a curve belonging to a linear complex coincide in pairs at the inflexions.*

If we put

$$
\begin{aligned}
x &= -t \\
y &= f - \tfrac{1}{2} t f' \\
z &= -f'
\end{aligned}
$$

so that we have the metric case, we find by elementary calculations that the curvature of a complex curve is

$$
\frac{1}{\rho} = \frac{f'''}{2(s')^3} \sqrt{t^2 + 4 + \overline{f'}^2}
$$

---

[5] B. Segre, "Sulle Curve le cui Tangenti Appartengono al Massimo numero di Complessi Lineari Independenti," *Memorie dell' Accademia dei Lincei*, series 6, vol. 2 (1928), pp. 578-592.

s, the arc length parameter. The torsion reduces to the simple form

$$\frac{1}{\sigma} = \frac{2}{t^2 + 4 + \overline{f'}^2}.$$

Thus, *At every real finite point the torsion* $1/\sigma$ *of a curve belonging to a linear complex is*

$$0 < 1/\sigma \leq \tfrac{1}{2}.^6$$

From this formula it is easy to deduce that the only curves of constant torsion belonging to a linear complex are helices on circular cylinders having $0Y$ as axis.

3. If in equations C, we put $\phi \equiv f^{2p+1} - at^{p+1} - bt^p = 0$ we have the curves

$$\begin{aligned} x_1 &= -(2p+1)t(at^{p+1} + bt^p)^{[2p/(2p+1)]} \\ x_2 &= \tfrac{1}{2}a(3p+1)t^{p+1} + \tfrac{1}{2}b(3p+2)t^p \\ x_3 &= -a(p+1)t^p - bpt^{p-1} \\ x_4 &= (2p+1)(at^{p+1} + bt^p)^{[2p/(2p+1)]}. \end{aligned}$$

These are of order $5p+2$ and of genus p. Each has a point of type $(p, p+1, p)$ at $(1, 0, 0, 0)$ and a point of type $(p+1, p, p+1)$ at $(0, 0, 1, 0)$. They lie on the ruled cubic surface

$$x_1(2bx_3x_4 - ax_1x_3 + ax_2x_4) - p(ax_1 - bx_4)(3x_1x_3 - 2x_2x_4) = 0.$$

If more generally we put $\phi \equiv bf^{KN} - at^{LN} - 1 = 0$ where $L > K$ and L is prime to K, we have the curves

$$\begin{aligned} x_1 &= -Kbtf^{KN-1} \\ x_2 &= \tfrac{1}{2}a(2K - L)t^{LN} + K \\ x_3 &= -Lat^{LN-1} \\ x_4 &= Kbf^{KN-1}. \end{aligned}$$

These are of genus $\dfrac{(LN-1)(KN-1) - (N-1)}{2}$ and of order KLN^2 except when $K = 1$ and $L = 2$, in which case the order becomes $2N^2 - N$. These curves have a singularity of type $[(L-K)N, (2K-L)N, (L-K)N]$ at $(0, 1, 0, 0)$ when $L < 2K$; they are non-singular when $L = 2K$, i. e. when $K = 1$, $L = 2$; and they have a singularity of type $[KN, (L-2K), KN]$ at $(0, 1, 0, 0)$ when $L > 2K$. For proper choices of K, L, N, these singularities can be made to represent any arbitrary self-dual singularity, say (s, t, s). Hence,

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[6] M. F. Egan, "The linear complex and a certain class of twisted curves," *Proceedings of the Royal Irish Academy*, vol. 29 (1911), pp. 33-72.

Curves can be found belonging to a linear complex and having any preassigned self-dual singularity counted any number of times.

4. Curves belonging to a linear complex have an $n-3 : n-3$ correspondence on them generated by the residual intersections of the curve and its osculating planes, and the points of contact of the osculating planes drawn to the curve from its various points. This correspondence is always composite, having at least one bilinear factor.[7]

If in equations C, we choose ϕ to be either

$$f^2 - 4t^3 + c^2 t = 0 \quad \text{or} \quad f^2 - 4t^3 + d^3 = 0$$

so that it is harmonic or equianharmonic respectively, the resulting curves are sextics of genus one, and the 3 : 3 correspondence defined on them by

$$\sum_{i=1}^{4} \frac{x_i(t)u_i(s)}{(t-s)^3} = 0 \qquad \begin{matrix} x, \text{ the parametric point equations} \\ u, \text{ the parametric plane equations} \end{matrix}$$

factors rationally into three bilinear factors. In each case these factors with the identity form a group of order four leaving the curve invariant. The points of each curve are thus arranged in tetrads such that the osculating plane at any one of them passes through the other three. The tetrads must consist then of collinear points, so that each curve has a regulus of quadrisecants, among which are three bitangents. These are the simplest irrational curves analogous to the rational quintic discussed by Egan.[8]

5. In 1877, Picard [9] showed that if a rational curve belonged to a complex it had to have $2(n-3)$ inflexions. In 1901 Grace [10] attempted to prove the sufficiency of this necessary condition. His argument was faulty however, and Egan [11] pointed out the error and gave an example of a sextic with six inflexions which did not belong to a linear complex. Gherardelli [12] elaborated this by showing that there were two classes of rational sextics, one of which consisted of curves belonging to a complex, the other of which consisted of

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[7] A. Terracini, "Sulla Riducibilita di Alcuna Particolari Corrispondenze Algebriche," *Rendiconti Circolo Matematico di Palermo*, vol. 56 (1923), pp. 112-143.

[8] M. F. Egan, *loc. cit.*

[9] E. Picard, *loc. cit.*

[10] J. H. Grace, "Curves in a linear complex," *Proceedings of the Cambridge Philological Society*, vol. 11 (1901), pp. 132-134.

[11] M. F. Egan, *loc. cit.*

[12] G. Gherardelli, "Le Sestiche Sghembe Razionali con Sei Flessi," *Rendiconti dell' Accademia dei Lincei*," series 6, vol. 2 (1930), pp. 173-179.

curves not belonging to a complex. He derived the first family of curves by projecting the rational normal sextic Γ_6 of S_6 onto an S_3 from a plane meeting six osculating planes of Γ_6, which plane did not lie on the V_5^2, fundamental for the polarity determined by Γ_6. The second family he derived by projecting Γ_6 from a plane meeting six osculating planes of Γ_6 and lying on the fundamental hyperquadric V_5^2. Curves of this second family he showed have their inflexions coplanar, and the osculating planes at the inflexions concurrent.

In the present paper the first class of sextics is obtained by sectioning the V_3^{12} of osculating planes to Γ_6 with an S_4 through two osculating rays of Γ_6 which are not coplanar, and the second class is obtained by sectioning the V_2^{10} of tangents to Γ_6 with an S_5 in general position, in each case interpreting the curve of intersection as a configuration on the fundamental hyperquadric V_4^2 of line geometry. These results will not be developed here.

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